

The BLG Theory in Light-Cone Superspace

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Abstract

The light-cone superspace version of the $d = 3, N = 8$ superconformal theory of Bagger, Lambert and Gustavsson (BLG) is obtained as a solution to constraints imposed by $OSp(2, 2|8)$ superalgebra. The Hamiltonian of the theory is shown to be a quadratic form of the dynamical supersymmetry transformation.

Keywords: Superspace; Light-cone; Superconformal Theories; Chern-Simons Theories.

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1 Introduction

The $d = 3, N = 8$ superconformal theory has recently been formulated covariantly by Bagger and Lambert [1], and Gustavsson [2], and its light-cone superspace formulation has been given in [3, 4]. In this paper, we will report on the use of algebraic techniques to construct this theory in light-cone superspace from its $OSp(2, 2|8)$ superconformal symmetry, using the same superfield (in one less dimension) that describes $d = 4, N = 4$ SuperYang-Mills.

The introduction of supersymmetry into quantum field theory has led to new restrictions in their quantum behavior. These effects are most spectacular in maximally supersymmetric theories. Although seemingly far from the real world, these theories constitute a starting point for the discussion of the rôle of symmetries in quantum field theories. It was realized long ago that the maximally supersymmetric $d = 4, N = 4$ Yang-Mills theory has unique properties, such as being finite in perturbation theory [5]. More recently it has been shown that $d = 4, N = 8$ supergravity also has remarkable properties in perturbation theory being finite at least up to four loops [6]. The underlying symmetry of the Yang-Mills theory is the full superconformal symmetry, $PSU(2, 2|4)$, while the symmetry in the supergravity case is the SuperPoincaré group times Cremmer and Julia's $E_{7(7)}$ [7] symmetry.

In a program that we have followed for quite some time [8] we have studied these theories and the corresponding ones in other space-time dimensions in Dirac's light-front form [9]. In this formalism we only use the physical degrees of freedom and the full SuperPoincaré algebra is non-linearly realized. It is the light-cone gauge formalism since we can reach the same result by the gauge choice that a light-cone component of the gauge fields be zero and by use of equations of motion to solve for the remaining unphysical degrees of freedom. We have found in this formalism a great similarity between the two classes of maximally supersymmetric theories and that they are each described by a superspace and a corresponding superfield that are universal.

The first superspace with *eight* complex Grassmann variables is used to describe maximally supersymmetric supergravity theories: $N = 1$ in $d = 11$, $N = 8$ in $d = 4$, $N = 16$ in $d = 3$, and so on. With a dimensionful coupling, these theories are not superconformal. They respect instead the non-compact and non-linear symmetries, $E_{7(7)}$ in $d = 4$, $E_{8(8)}$ in $d = 3$, etc., with light-cone superspace formulation written in terms of the same constrained chiral superfield [10, 11].

The second superspace with *four* complex Grassmann variables is equally rich. It houses theories with maximal superconformal symmetry in $d = 6, 5, 4$ and 3 dimensions, as well as other maximally supersymmetric gauge theories such as $N = 1, d = 10$ SuperYang-Mills. It has already been shown [12] how the fully interacting $d = 4, N = 4$ SuperYang-Mills theory [13] can be determined by requiring $PSU(2, 2|4)$ superconformal symmetry on a constrained chiral superfield in this light-cone superspace. In this paper, we will present a similar analysis of the

$d = 3, N = 8$ superconformal theory. This will be an alternative way to find the BLG-theory, which will open up new venues to investigate the model and to find its limitations and possible extensions.

In the light-cone formulation (on the light front), symmetries split into kinematical and dynamical ones. Kinematical symmetries are linearly realized, while dynamical ones contain a linear term (free theory), and terms non-linear in the (super)fields. In superconformal theories, dynamical supersymmetries suffice to *completely* determine the theory algebraically. Our technique is to use algebraic consistency to find all possible non-linear realizations of the algebra on the chiral superfields.

2 The $N = 4$ Chiral Superfield

Introduce the usual light-cone variables in d spacetime dimensions, $x^\pm = (x^0 \pm x^{d-1})/\sqrt{2}$, and their derivatives $\partial^\pm = (\partial^0 \pm \partial^{d-1})/\sqrt{2}$ satisfying $[\partial^+, x^-] = [\partial^-, x^+] = -1$, with the metric $\eta^{\mu\nu} = (-, +, \dots, +)$. The $N = 4$ superspace contains four complex anticommuting Grassmann variables, θ^m , ($m = 1, \dots, 4$) and their conjugates $\bar{\theta}_m$,

Fundamental to this superspace are the chiral superfields

$$\begin{aligned} \varphi^a(y) = & \frac{1}{\partial^+} A^a(y) + \frac{i}{\sqrt{2}} \theta^m \theta^n \bar{C}_{mn}^a(y) + \frac{1}{12} \theta^m \theta^n \theta^p \theta^q \varepsilon_{mnpq} \partial^+ \bar{A}^a(y) \\ & + \frac{i}{\partial^+} \theta^m \bar{\chi}_m^a(y) + \frac{\sqrt{2}}{6} \theta^m \theta^n \theta^p \varepsilon_{mnpq} \chi^{qa}(y), \end{aligned} \quad (2.1)$$

where a is a taxonomic index and $y = (x_1, \dots, x_{d-2}, x^+, x^- - i\theta^m \bar{\theta}_m/\sqrt{2})$ are chiral coordinates¹. The superfields are chiral

$$d^m \varphi^a(y) = 0, \quad (2.2)$$

and obey the “inside-out” constraint

$$\bar{d}_m \bar{d}_n \varphi^a = \frac{1}{2} \varepsilon_{mnpq} d^p d^q \bar{\varphi}^a, \quad (2.3)$$

where the chiral derivatives

$$d^m = -\frac{\partial}{\partial \theta_m} - \frac{i}{\sqrt{2}} \theta^m \partial^+; \quad \bar{d}_n = \frac{\partial}{\partial \bar{\theta}_n} + \frac{i}{\sqrt{2}} \bar{\theta}_n \partial^+, \quad (2.4)$$

satisfy

$$\{d^m, \bar{d}_n\} = -i\sqrt{2} \delta^m{}_n \partial^+. \quad (2.5)$$

The component fields A , \bar{A} , and \bar{C}_{mn} represent eight bosons; χ^m and $\bar{\chi}_m$ are the eight fermions.

¹We will take $x^+ = 0$ for the light-front surface with respect to which the generators of $OSp(2, 2|8)$ will be defined.

3 The Superconformal Algebra in $d = 3$: $OSp(2, 2|8)$

In $d = 6, 5, 4$, and 3 , chiral superfields (2.1) form a linear representation of the conformal superalgebras in these dimensions. In $d = 4$, they can be used to describe the interacting $N = 4$ SuperYang-Mills theory, with a representation of the $PSU(2, 2|4)$ superalgebra non-linear in the superfield. The same superfields, in $d = 3$, can be used to describe the interacting $N = 8$ SuperChern-Simons (BLG) theory, with the superalgebra $OSp(2, 2|8)$ realized nonlinearly. This superalgebra has the following bosonic subalgebra

$$SO(8) \times Sp(2, 2) \subset OSp(2, 2|8) ,$$

where $SO(8)$ is the R -symmetry, and $Sp(2, 2) \sim SO(3, 2)$ is the conformal group in three dimensions. Below we will give a representation of this superalgebra in terms of operators corresponding to a free (non-interacting) theory. (See also Appendix A.)

3.1 R -symmetries

The action of the R -symmetry on the chiral superfield is expressed in terms of the operators (the kinematical supersymmetry generators)

$$q^m = -\frac{\partial}{\partial\bar{\theta}_m} + \frac{i}{\sqrt{2}}\theta^m\partial^+ ; \quad \bar{q}_n = \frac{\partial}{\partial\theta^n} - \frac{i}{\sqrt{2}}\bar{\theta}_n\partial^+ , \quad (3.1)$$

which satisfy

$$\{q^m, \bar{q}_n\} = i\sqrt{2}\delta^m{}_n\partial^+ . \quad (3.2)$$

They do not affect chirality since they anticommute with the chiral derivatives.

The $SO(8)$ R -symmetry is written as $SO(6) \times SO(2) \sim SU(4) \times U(1)$ transformations, with generators $T^m{}_n$, and T ,

$$\begin{aligned} \delta_{SU(4)}\varphi^a &= \omega^m{}_n T^n{}_m \varphi^a = \omega^m{}_n \frac{i}{\sqrt{2}} \left(q^n \bar{q}_m - \frac{1}{4} \delta_m^n q^k \bar{q}_k \right) \frac{1}{\partial^+} \varphi^a , \\ \delta_{U(1)}\varphi^a &= \omega T \varphi^a = \omega \frac{i}{4\sqrt{2}} (q^m \bar{q}_m - \bar{q}_m q^m) \frac{1}{\partial^+} \varphi^a , \end{aligned} \quad (3.3)$$

together with the coset transformations, with generators T^{mn} , and \bar{T}_{mn} ,

$$\begin{aligned} \delta_{\overline{\text{coset}}}\varphi^a &= \omega^{mn} \bar{T}_{mn} \varphi^a = \omega^{mn} \frac{i}{\sqrt{2}} \bar{q}_m \bar{q}_n \frac{1}{\partial^+} \varphi^a ; \\ \delta_{\text{coset}}\varphi^a &= \bar{\omega}_{mn} T^{mn} \varphi^a = \bar{\omega}_{mn} \frac{i}{\sqrt{2}} q^m q^n \frac{1}{\partial^+} \varphi^a , \end{aligned} \quad (3.4)$$

completing the full $SO(8) \supset SO(6) \times SO(2)$. All R -symmetry generators are kinematical.

3.2 Superconformal Symmetries

Space-time generators are either kinematical or dynamical. Kinematical generators operate within the initial surface, while the dynamical ones act transversely to the initial surface, and define the dynamics. The kinematical generators are the same in free and interacting theories, and induce changes linear in the fields. The dynamical generators, on the other hand, contain a part linear in the (super)fields for the free theory, as well as terms which are non-linear in the (super)fields, accounting for the interactions.

In light-cone notation, the ten generators of the conformal group in three dimensions are given by

$$\begin{aligned} \text{Lorentz Group : } & J^{+-}, J^+ ; \quad \mathcal{J}^- \\ \text{Translations : } & P, P^+ ; \quad \mathcal{P}^- \\ \text{Dilatation : } & D \\ \text{Conformal : } & K, K^+ ; \quad \mathcal{K}^- \end{aligned}$$

with the dynamical generators written in calligraphic letters. Note that J^{+-} and K^+ , K and D are kinematical *only* at $x^+ = 0$ (cf. [8]). The supersymmetry and superconformal (or conformal supersymmetry) generators, which complete the superconformal algebra, also split into kinematical and dynamical generators

$$\begin{aligned} \text{Supersymmetry : } & q^m, \bar{q}_m ; \quad \mathcal{Q}^m, \bar{\mathcal{Q}}_m \\ \text{Superconformal : } & s^m, \bar{s}_m ; \quad \mathcal{S}^m, \bar{\mathcal{S}}_m . \end{aligned}$$

3.3 Kinematical Transformations

The kinematical conformal group transformations are given by

$$\begin{aligned} \delta_{P^+} \varphi^a &= -i \partial^+ \varphi^a ; \quad \delta_P \varphi^a = -i \partial \varphi^a ; \\ \delta_{J^+} \varphi^a &= ix \partial^+ \varphi^a ; \quad \delta_{J^{+-}} \varphi^a = i(\mathcal{A} + \frac{x}{2} \partial + \frac{1}{2}) \varphi^a ; \\ \delta_D \varphi^a &= i(\mathcal{A} - \frac{x}{2} \partial) \varphi^a ; \quad \delta_K \varphi^a = 2ix \mathcal{A} \varphi^a ; \quad \delta_{K^+} \varphi^a = ix^2 \partial^+ \varphi^a , \end{aligned} \quad (3.5)$$

where ∂ is the derivative with respect to the lone transverse variable x in the superfield, and

$$\mathcal{A} \equiv x^- \partial^+ - \frac{x}{2} \partial - \frac{1}{2} \mathcal{N} + \frac{1}{2} ; \quad \mathcal{N} \equiv \theta^m \frac{\partial}{\partial \theta^m} + \bar{\theta}_m \frac{\partial}{\partial \bar{\theta}_m} . \quad (3.6)$$

The kinematical (spectrum-generating) supersymmetries are

$$\delta_{\varepsilon \bar{q}} \varphi^a = \varepsilon^m \bar{q}_m \varphi^a ; \quad \delta_{\bar{\varepsilon} q} \varphi^a = \bar{\varepsilon}_m q^m \varphi^a , \quad (3.7)$$

and the kinematical superconformal transformations are

$$\delta_{\varepsilon\bar{s}} \varphi^a = -ix\varepsilon^m \bar{q}_m \varphi^a ; \quad \delta_{\bar{\varepsilon}s} \varphi^a = ix\bar{\varepsilon}_m q^m \varphi^a . \quad (3.8)$$

where ε^m and $\bar{\varepsilon}_m$ are anticommuting parameters.

3.4 Free Dynamical Transformations

A distinguishing feature of superconformal theories is that *all* dynamical generators are determined by commutations from the dynamical supersymmetry generators. Starting from the *free* dynamical supersymmetry transformations²,

$$\delta_{\varepsilon\bar{Q}}^{free} \varphi^a = \frac{1}{\sqrt{2}} \varepsilon^m \bar{q}_m \frac{\partial}{\partial^+} \varphi^a , \quad \delta_{\bar{\varepsilon}Q}^{free} \varphi^a = \frac{1}{\sqrt{2}} \bar{\varepsilon}_m q^m \frac{\partial}{\partial^+} \varphi^a , \quad (3.9)$$

we use the algebra

$$\begin{aligned} [\delta_{\varepsilon\bar{Q}}, \delta_{\bar{\varepsilon}Q}] \varphi^a &= \sqrt{2} \bar{\varepsilon}_m \varepsilon^m \delta_{P-} \varphi^a \quad \rightarrow \quad \delta_{P-} \varphi^a , \\ [\delta_K, \delta_{P-}] \varphi^a &= 2i \delta_{J-} \varphi^a \quad \rightarrow \quad \delta_{J-} \varphi^a , \\ [\delta_K, \delta_{J-}] \varphi^a &= -i \delta_{K-} \varphi^a \quad \rightarrow \quad \delta_{K-} \varphi^a , \\ [\delta_K, \delta_{\varepsilon\bar{Q}}] \varphi^a &= \sqrt{2} \delta_{\varepsilon\bar{S}} \varphi^a \quad \rightarrow \quad \delta_{\varepsilon\bar{S}} \varphi^a , \\ [\delta_K, \delta_{\bar{\varepsilon}Q}] \varphi^a &= -\sqrt{2} \delta_{\bar{\varepsilon}S} \varphi^a \quad \rightarrow \quad \delta_{\bar{\varepsilon}S} \varphi^a , \end{aligned} \quad (3.10)$$

to obtain the remaining dynamical transformations,

$$\begin{aligned} \text{“Time” } (x^+) \text{ Translation : } \delta_{P-}^{free} \varphi^a &= -i \frac{\partial^2}{2\partial^+} \varphi^a , \\ \text{Lorentz Boost : } \delta_{J-}^{free} \varphi^a &= -i \frac{\partial}{\partial^+} \mathcal{A} \varphi^a , \\ \text{Conformal Boost : } \delta_{K-}^{free} \varphi^a &= 2i \frac{1}{\partial^+} \mathcal{A} (\mathcal{A} - \frac{1}{2}) \varphi^a , \\ \text{Superconformal : } \delta_{\varepsilon\bar{S}}^{free} \varphi^a &= i \varepsilon^m \bar{q}_m \frac{1}{\partial^+} \mathcal{A} \varphi^a , \\ \delta_{\bar{\varepsilon}S}^{free} \varphi^a &= -i \bar{\varepsilon}_m q^m \frac{1}{\partial^+} \mathcal{A} \varphi^a . \end{aligned} \quad (3.11)$$

This representation of the dynamical generators is valid in the free theory, and needs to be augmented in the interacting theory. Together with the kinematical generators, they satisfy the $OSp(2, 2|8)$ algebra, whose light-cone commutation relations appear in Appendix A.

²For emphasis, we write *dynamical* transformations with a bold δ .

4 Interactions

In the interacting theory, the dynamical generators acquire contributions nonlinear in the superfields. To specify the full theory, we need only find these contributions to the dynamical supersymmetry generators. All other dynamical generators follow from the algebra by commutations.

4.1 Kinematical Constraints

The dynamical supersymmetries consist of two parts:

$$\delta_{\varepsilon\bar{Q}}\varphi^a = \delta_{\varepsilon\bar{Q}}^{free}\varphi^a + \delta_{\varepsilon\bar{Q}}^{int}\varphi^a, \quad \delta_{\bar{\varepsilon}Q}\varphi^a = \delta_{\bar{\varepsilon}Q}^{free}\varphi^a + \delta_{\bar{\varepsilon}Q}^{int}\varphi^a. \quad (4.1)$$

The forms of $\delta_{\varepsilon\bar{Q}}^{int}\varphi^a$ and $\delta_{\bar{\varepsilon}Q}^{int}\varphi^a$ are highly restricted by the following ten algebraic constraints [14]:

(i) Chirality: the transformations should be chiral, that is,

$$d^m(\delta_{\varepsilon\bar{Q}}^{int}\varphi^a) = d^m(\delta_{\bar{\varepsilon}Q}^{int}\varphi^a) = 0, \quad (4.2)$$

and satisfy the inside-out constraint,

$$\delta_{\bar{\varepsilon}Q}^{int}\varphi^a = \frac{d^{[4]}}{2\partial^{+2}} (\delta_{\varepsilon\bar{Q}}^{int}\varphi^a)^*, \quad (4.3)$$

where $d^{[4]} \equiv d^1 d^2 d^3 d^4$.

(ii) Both are independent of x^- , since

$$[\delta_{P+}, \delta_{\varepsilon\bar{Q}}]\varphi^a = [\delta_{P+}, \delta_{\bar{\varepsilon}Q}]\varphi^a = 0. \quad (4.4)$$

(iii) Both are also independent of x , as

$$[\delta_P, \delta_{\varepsilon\bar{Q}}]\varphi^a = [\delta_P, \delta_{\bar{\varepsilon}Q}]\varphi^a = 0. \quad (4.5)$$

(iv) Neither have transverse derivatives ∂ : from

$$[\delta_{J+}, \delta_{\bar{\varepsilon}Q}]\varphi^a = \frac{i}{\sqrt{2}}\delta_{\bar{\varepsilon}q}\varphi^a, \quad [\delta_{J+}, \delta_{\varepsilon\bar{Q}}]\varphi^a = \frac{i}{\sqrt{2}}\delta_{\varepsilon\bar{q}}\varphi^a, \quad (4.6)$$

it follows that

$$[\delta_{J+}, \delta_{\bar{\varepsilon}Q}^{int}]\varphi^a = [\delta_{J+}, \delta_{\varepsilon\bar{Q}}^{int}]\varphi^a = 0. \quad (4.7)$$

(v) From

$$[\delta_{\bar{\varepsilon}q}, \boldsymbol{\delta}_{\varepsilon\bar{Q}}] \varphi^a = -\bar{\varepsilon}_m \varepsilon^m \delta_P \varphi^a, \quad [\delta_{\varepsilon\bar{q}}, \boldsymbol{\delta}_{\bar{\varepsilon}Q}] \varphi^a = \bar{\varepsilon}_m \varepsilon^m \delta_P \varphi^a, \quad (4.8)$$

we deduce that

$$[\delta_{\bar{\varepsilon}q}, \boldsymbol{\delta}_{\varepsilon\bar{Q}}^{int}] \varphi^a = [\delta_{\varepsilon\bar{q}}, \boldsymbol{\delta}_{\bar{\varepsilon}Q}^{int}] \varphi^a = 0. \quad (4.9)$$

(vi) Proper transformation under J^{+-} require

$$[\delta_{J^{+-}}, \boldsymbol{\delta}_{\varepsilon\bar{Q}}^{int}] \varphi^a = \frac{i}{2} \boldsymbol{\delta}_{\varepsilon\bar{Q}}^{int} \varphi^a, \quad [\delta_{J^{+-}}, \boldsymbol{\delta}_{\bar{\varepsilon}Q}^{int}] \varphi^a = \frac{i}{2} \boldsymbol{\delta}_{\bar{\varepsilon}Q}^{int} \varphi^a. \quad (4.10)$$

(vii) Proper transformations under D require

$$[\delta_D, \boldsymbol{\delta}_{\varepsilon\bar{Q}}^{int}] \varphi^a = -\frac{i}{2} \boldsymbol{\delta}_{\varepsilon\bar{Q}}^{int} \varphi^a, \quad [\delta_D, \boldsymbol{\delta}_{\bar{\varepsilon}Q}^{int}] \varphi^a = -\frac{i}{2} \boldsymbol{\delta}_{\bar{\varepsilon}Q}^{int} \varphi^a. \quad (4.11)$$

(viii) They have opposite $U(1)$ R -charge,

$$[\delta_{U(1)}, \boldsymbol{\delta}_{\varepsilon\bar{Q}}^{int}] \varphi^a = -\frac{1}{2} \boldsymbol{\delta}_{\varepsilon\bar{Q}}^{int} \varphi^a, \quad [\delta_{U(1)}, \boldsymbol{\delta}_{\bar{\varepsilon}Q}^{int}] \varphi^a = \frac{1}{2} \boldsymbol{\delta}_{\bar{\varepsilon}Q}^{int} \varphi^a. \quad (4.12)$$

(ix) The eight interacting supersymmetries must also transform as an $SO(8)$ vector, that is, with $\bar{\varepsilon}'_m = 2\bar{\omega}_{mn} \varepsilon^n$,

$$[\delta_{\overline{\text{coset}}}, \boldsymbol{\delta}_{\varepsilon\bar{Q}}] \varphi^a = 0, \quad [\delta_{\text{coset}}, \boldsymbol{\delta}_{\varepsilon\bar{Q}}] \varphi^a = \boldsymbol{\delta}_{\varepsilon'\bar{Q}} \varphi^a. \quad (4.13)$$

Similarly, with $\varepsilon'^m = 2\omega^{mn} \bar{\varepsilon}_n$,

$$[\delta_{\overline{\text{coset}}}, \boldsymbol{\delta}_{\bar{\varepsilon}Q}] \varphi^a = \boldsymbol{\delta}_{\varepsilon'\bar{Q}} \varphi^a, \quad [\delta_{\text{coset}}, \boldsymbol{\delta}_{\bar{\varepsilon}Q}] \varphi^a = 0. \quad (4.14)$$

(x) Both $\boldsymbol{\delta}_{\varepsilon\bar{Q}}^{int} \varphi^a$ and $\boldsymbol{\delta}_{\bar{\varepsilon}Q}^{int} \varphi^a$ are cubic powers of the superfields.

In three dimensions, canonical Bose fields have mass dimension of one-half, so that the chiral superfield has half-odd integer canonical dimension. Since we are looking for a conformal theory with no dimensionful parameters, $\boldsymbol{\delta}_{\varepsilon\bar{Q}}^{int} \varphi^a$ and $\boldsymbol{\delta}_{\bar{\varepsilon}Q}^{int} \varphi^a$ must then both be odd powers of superfields, assuming integer power of derivatives. To allow for three or more superfields, the theory must contain a tensor with at least four indices, f_{bcd}^a ³.

To see that it is only cubic, we form the combination

³In $d = 4$, similar considerations suggested a tensor with three indices, f_{bc}^a , which turned out to be the structure functions of the gauge algebra.

$$\Delta \equiv J^{+-} - D = i \left(x\partial + \frac{1}{2} \right), \quad (4.15)$$

where $x\partial$ counts the number of transverse variables, and the constant counts the number of superfields. Since $\delta_{\epsilon\overline{Q}}^{int}\varphi^a$ does not contain any explicit transverse variables, and assuming that it contains products of n_φ superfields, it follows that

$$[\delta_\Delta, \delta_{\epsilon\overline{Q}}^{int}]\varphi^a = \frac{i}{2}(n_\varphi - 1)\delta_{\epsilon\overline{Q}}^{int}\varphi^a. \quad (4.16)$$

On the other hand, the algebra requires

$$[\delta_\Delta, \delta_{\epsilon\overline{Q}}^{int}]\varphi^a = i\delta_{\epsilon\overline{Q}}^{int}\varphi^a, \quad (4.17)$$

where $\delta_\Delta\varphi^a = \Delta\varphi^a$. These agree when $n_\varphi = 3$, limiting the interacting supersymmetry to a cubic form.

These ten requirements limit the possible forms of the dynamical supersymmetries.

4.2 Chiral Engineering

The construction of chiral polynomials in the superfields is facilitated by the introduction of the coherent state operators [10, 11]

$$E_\eta = e^{\eta \cdot \widehat{\overline{d}}}, \quad (4.18)$$

where the hat denotes division by ∂^+ , $\widehat{\overline{d}}_m \equiv \overline{d}_m/\partial^+$, and η^m are arbitrary Grassmann parameters. Since

$$d^m (E_\eta \varphi^a) = i\sqrt{2} \eta^m (E_\eta \varphi^a), \quad (4.19)$$

$E_\eta \varphi^a$ are eigenstates of the chiral derivatives. It follows that the quadratic combination

$$(E_\eta \partial^{+B} \varphi^b) (E_{-\eta} \partial^{+C} \varphi^c), \quad (4.20)$$

is manifestly chiral. The *nested form*

$$(E_\eta \partial^{+B} \varphi^b) E_{-\eta} \frac{1}{\partial^{+M}} ((E_\zeta \partial^{+C} \varphi^c) (E_{-\zeta} \partial^{+D} \varphi^d)), \quad (4.21)$$

is also chiral, and can be used to generate chiral cubic polynomials in the superfields, the coefficients in the series expansion in the independent Grassmann parameters η and ζ .

4.3 Even and Odd Ansätze

To construct the interaction part of the dynamical supersymmetry, we introduce the supersymmetry parameters in the nested Ansatz through the combinations

$$E_\varepsilon = e^{\varepsilon \cdot \widehat{\bar{q}}}, \quad E_{\bar{\varepsilon}} = e^{\bar{\varepsilon} \cdot \widehat{q}}, \quad (4.22)$$

which naturally allow to satisfy requirement (v), without affecting chirality. This leads us to write the dynamical supersymmetries as a sum of nested ansätze of the form

$$\begin{aligned} \delta_{\varepsilon \bar{\mathcal{Q}}}^{int} \varphi^a &\sim \frac{f^a_{bcd}}{\partial^{+A_\alpha}} \left((E_\varepsilon E_\eta \partial^{+B_\alpha} \varphi^b) E_{-\varepsilon} E_{-\eta} \frac{1}{\partial^{+M_\alpha}} ((E_\zeta \partial^{+C_\alpha} \varphi^c) (E_{-\zeta} \partial^{+D_\alpha} \varphi^d)) \right), \\ &\equiv \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)}, \end{aligned} \quad (4.23)$$

keeping only the first order in the supersymmetry parameters ε^m . We have to allow for a non-trivial sum over α . f^a_{bcd} and the exponents $A_\alpha, B_\alpha, M_\alpha, C_\alpha, D_\alpha$ have yet to be determined. It is convenient to introduce *insertion operators* \mathcal{U}_i ($i = 1, 2, 3, 4$), whose action is defined by

$$\begin{aligned} \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)} &= (E_\varepsilon \mathcal{U}_1) (E_{-\varepsilon} \mathcal{U}_2) \mathcal{K}_\alpha^{a(0, \eta, \zeta)} = (E_\zeta \mathcal{U}_3) (E_{-\zeta} \mathcal{U}_4) \mathcal{K}_\alpha^{a(\varepsilon, \eta, 0)} \\ &= (E_\varepsilon E_\eta, E_{-\varepsilon} E_{-\eta} (E_\zeta, E_{-\zeta})) \mathcal{K}_\alpha^{a(0, 0, 0)}. \end{aligned} \quad (4.24)$$

We will often use the useful (, (,)) notation when we have an operator which makes multiple insertions.

For this Ansatz, we find that (see Appendix B for more details)

- Chirality (i) is manifest since the \bar{q}_n anticommute with the chiral derivatives. The inside-out constraint (4.3) will be checked below.
- (ii), (iii), (iv), and (v) are clearly satisfied.
- The proper transformation under J^{+-} , (vi), together with the $U(1)$ condition, (viii), restricts the number of ∂^+ derivatives to four,

$$-A_\alpha + B_\alpha - M_\alpha + C_\alpha + D_\alpha = 4, \quad (4.25)$$

which reproduces the correct dimension.

- The correct $U(1)$ *R*-charge, (viii), requires after some computation

$$\left(\eta^m \frac{\partial}{\partial \eta^m} + \zeta^m \frac{\partial}{\partial \zeta^m} - 4 \right) \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)} = 0, \quad (4.26)$$

so that only the coefficients of the terms *quartic* in η and ζ ,

$$\eta^4, \quad \eta^3\zeta, \quad \eta^2\zeta^2, \quad \eta\zeta^3, \quad \zeta^4 \quad (4.27)$$

need to be considered.

- We find

$$[\delta_{\text{coset}}, \delta_{\varepsilon \overline{\mathcal{Q}}}^{\text{int}}] \varphi^a = \frac{1}{\sqrt{2}} \omega^{mn} \sum \left(\frac{\partial}{\partial \eta^m} \frac{\partial}{\partial \eta^n} \mathcal{S} + \frac{\partial}{\partial \zeta^m} \frac{\partial}{\partial \zeta^n} \mathcal{T} \right) \mathcal{K}_\alpha^{a(\epsilon, \eta, \zeta)}, \quad (4.28)$$

where \mathcal{S} and \mathcal{T} are multiple insertion operators defined by

$$\mathcal{S} \equiv \frac{1}{\partial^+} (\partial^+, \partial^+(1, 1)), \quad \mathcal{T} \equiv (1, \frac{1}{\partial^+} (\partial^+, \partial^+)). \quad (4.29)$$

The right hand side has to vanish. Because of the appearance of *double* η - and ζ -derivatives in this expression, the “even” and “odd” sets,

$$(\eta^4, \quad \eta^2\zeta^2, \quad \zeta^4) \quad \text{and} \quad (\eta^3\zeta, \quad \eta\zeta^3), \quad (4.30)$$

do not mix. This splits our Ansatz into two:

- the even Ansatz⁴

$$\delta_{\varepsilon \overline{\mathcal{Q}}}^{\text{int}, \text{even}} \varphi^a = \frac{1}{\sqrt{2}} \sum_{\text{even}} \mathcal{K}_\alpha^{a(\epsilon, \eta, \zeta)} \Big|_{\eta=\zeta=0; \text{ linear in } \varepsilon}, \quad (4.31)$$

where the sum stands for the operator

$$\sum_{\text{even}} \equiv \sum_{\alpha=0, \pm 1} (-1)^\alpha \frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \quad (4.32)$$

with

$$\frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \equiv \frac{\epsilon^{i_1 \dots i_4}}{(2-2\alpha)!(2+2\alpha)!} \frac{\partial}{\partial \eta^{i_1 \dots i_{2-2\alpha}}} \frac{\partial}{\partial \zeta^{i_{3-2\alpha} \dots i_4}} \quad (4.33)$$

- and the odd Ansatz

$$\delta_{\varepsilon \overline{\mathcal{Q}}}^{\text{int}, \text{odd}} \varphi^a = \frac{1}{\sqrt{2}} \sum_{\text{odd}} \mathcal{K}_\alpha^{a(\epsilon, \eta, \zeta)} \Big|_{\eta=\zeta=0; \text{ linear in } \varepsilon}, \quad (4.34)$$

where

$$\sum_{\text{odd}} \equiv \sum_{\alpha=\pm 1/2} (-1)^{\alpha+\frac{1}{2}} \frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \quad (4.35)$$

⁴The nested form (4.22) was inspired by the structure of the $O(\kappa^2)$ part of the dynamical supersymmetry in $d=3$, $N=16$ supergravity, as given in equation (4.14) in [11] (the right hand side of that equation should include an extra factor $(-1)^c/(4+2c)!$). Note that our even Ansatz (4.31) is its direct analog.

We then find that the first constraint in (4.13) is satisfied for both ansätze provided

$$\mathcal{K}_{\alpha+1}^a = \mathcal{S}^{-1} \mathcal{T} \mathcal{K}_\alpha^a = \partial^+ \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+2}} (\partial^+, \partial^+) \right) \mathcal{K}_\alpha^a, \quad (4.36)$$

which yields a recursion relation for the powers of ∂^+

$$\begin{aligned} A_{\alpha+1} &= A_\alpha - 1, & B_{\alpha+1} &= B_\alpha - 1, & M_{\alpha+1} &= M_\alpha + 2 \\ C_{\alpha+1} &= C_\alpha + 1, & D_{\alpha+1} &= D_\alpha + 1. \end{aligned} \quad (4.37)$$

- To verify the second constraint in (4.13), we find that

$$\begin{aligned} [\delta_{\text{coset}}, \delta_{\bar{\varepsilon}\bar{\mathcal{Q}}}^{\text{int}}] \varphi^a &= 2i\bar{\omega}_{mn} \varepsilon^n \sum \eta^m \mathcal{S}^{-1} \mathcal{K}_\alpha^{a(0,\eta,\zeta)} \\ &\quad - i\bar{\omega}_{mn} \sum (\eta^m \eta^n \mathcal{S}^{-1} + \zeta^m \zeta^n \mathcal{T}^{-1}) \mathcal{K}_\alpha^{a(\epsilon,\eta,\zeta)}. \end{aligned} \quad (4.38)$$

The sum in the second line vanishes in both the even and odd cases, thanks to the recursion relation (4.36). The second constraint in (4.13) is then satisfied provided

$$\delta_{\bar{\varepsilon}\bar{\mathcal{Q}}}^{\text{int}} \varphi^a = i\bar{\varepsilon}_m \sum \eta^m \mathcal{S}^{-1} \mathcal{K}_\alpha^{a(0,\eta,\zeta)} = \frac{1}{\sqrt{2}} \sum \mathcal{K}_\alpha^{a(\bar{\varepsilon},\eta,\zeta)} \Big|_{\text{linear in } \bar{\varepsilon}}. \quad (4.39)$$

We have verified that the inside-out constraint (4.3) is in agreement with (4.39) in both the even and odd case, thanks again to the recursion relation (4.37). (See Appendix B for more details.)

To summarize, we have found that both the even Ansatz (4.31) and the odd Ansatz (4.34), with the exponents $A_\alpha, B_\alpha, M_\alpha, C_\alpha, D_\alpha$ satisfying the dimensional constraint (4.25) and the recursion relation (4.37), are solutions to the chirality, inside-out and all the kinematical constraints. Next, we turn to satisfying the dynamical constraints.

4.4 $\delta_{\mathcal{P}-} \varphi^a$, $\delta_{\varepsilon\bar{\mathcal{Q}}} \varphi^a$, and $\delta_{\mathcal{K}-} \varphi^a$

Having found ansätze for $\delta_{\varepsilon\bar{\mathcal{Q}}}^{\text{int}} \varphi^a$ and $\delta_{\bar{\varepsilon}\bar{\mathcal{Q}}}^{\text{int}} \varphi^a$ that satisfy all the kinematical constraints, we can use (3.10) to calculate the remaining dynamical transformations which will automatically satisfy their own kinematical constraints thanks to the Jacobi identities. There is a subtlety, however, in the calculation of the Hamiltonian shift $\delta_{\mathcal{P}-} \varphi^a$ via

$$[\delta_{\varepsilon\bar{\mathcal{Q}}}^{\text{free}} + \delta_{\varepsilon\bar{\mathcal{Q}}}^{\text{int}}, \delta_{\bar{\varepsilon}\bar{\mathcal{Q}}}^{\text{free}} + \delta_{\bar{\varepsilon}\bar{\mathcal{Q}}}^{\text{int}}] \varphi^a = \sqrt{2} \bar{\varepsilon}_m \varepsilon^m \delta_{\mathcal{P}-} \varphi^a, \quad (4.40)$$

as one should verify that the “off-diagonal” terms $\bar{\varepsilon}_m \varepsilon^n$, with $m \neq n$, all cancel. The interaction part of the dynamical supersymmetry is linear in f^a_{bcd} , while the Hamiltonian shift has both linear, $\delta_{\mathcal{P}-}^{(1)} \varphi^a$, and quadratic, $\delta_{\mathcal{P}-}^{(2)} \varphi^a$, parts:

$$\delta_{\mathcal{P}-}^{\text{int}} \varphi^a = \delta_{\mathcal{P}-}^{(1)} \varphi^a + \delta_{\mathcal{P}-}^{(2)} \varphi^a. \quad (4.41)$$

The $O(f)$ part is then determined by

$$[\delta_{\varepsilon\overline{Q}}^{free}, \delta_{\overline{\varepsilon}Q}^{int}] \varphi^a + [\delta_{\varepsilon\overline{Q}}^{int}, \delta_{\overline{\varepsilon}Q}^{free}] \varphi^a = \sqrt{2} \overline{\epsilon}_m \varepsilon^m \delta_{P-}^{(1)} \varphi^a, \quad (4.42)$$

and we verified (see Appendix C) that the off-diagonal terms at this order cancel for both the even and odd ansätze. In the odd case, the result for the Hamiltonian shift is

$$\delta_{P-}^{(1) odd} \varphi^a = -\frac{i}{2} \frac{\partial}{\partial r} \left(\sum_{\text{odd}} K_\alpha^{a[r,1]} + \sum_{\text{even}} K_{\alpha+\frac{1}{2}}^{a[1,r]} \right) \Big|_{\eta=\zeta=r=0}, \quad (4.43)$$

where

$$K_\alpha^{a[r,1]} \equiv (E_r \mathcal{U}_1)(E_{-r} \mathcal{U}_2) \mathcal{K}_\alpha^{a(0,\eta,\zeta)}, \quad K_\alpha^{a[1,r]} \equiv (E_r \mathcal{U}_3)(E_{-r} \mathcal{U}_4) \mathcal{K}_\alpha^{a(0,\eta,\zeta)}, \quad (4.44)$$

introducing the ∂ -exponential,

$$E_r = e^{r\widehat{\partial}}, \quad (4.45)$$

where $\widehat{\partial} = \partial/\partial^+$ (∂ is the transverse derivative) and r is a dimensionless parameter. The result in the even case is similar and can be obtained from (4.43) by the following substitutions

$$\sum_{\text{odd}} \rightarrow \sum_{\text{even}}, \quad \sum_{\text{even}} \rightarrow -\sum_{\text{odd}}. \quad (4.46)$$

The dynamical superconformal transformations are easily computed, using the transverse kinematical conformal operator K , which acts as a ladder operator,

$$[\delta_K, \delta_{\varepsilon\overline{Q}}] \varphi^a = \sqrt{2} \delta_{\varepsilon\overline{S}} \varphi^a, \quad [\delta_K, \delta_{\overline{\varepsilon}Q}] \varphi^a = -\sqrt{2} \delta_{\overline{\varepsilon}S} \varphi^a, \quad (4.47)$$

which yields, using $K = 2ix\mathcal{A}$,

$$\delta_{\overline{\varepsilon}S}^{int} \varphi^a = \frac{i}{\sqrt{2}} x \delta_{\overline{\varepsilon}Q}^{int} \varphi^a, \quad \delta_{\varepsilon\overline{S}}^{int} \varphi^a = -\frac{i}{\sqrt{2}} x \delta_{\varepsilon\overline{Q}}^{int} \varphi^a. \quad (4.48)$$

Then $\delta_{K-} \varphi^a$ follows from

$$[\delta_{\varepsilon\overline{S}}, \delta_{\overline{\varepsilon}S}] \varphi^a = -\frac{1}{\sqrt{2}} \overline{\epsilon}_m \varepsilon^m \delta_{K-} \varphi^a. \quad (4.49)$$

The cancellation of its off-diagonal terms follow from

$$[\delta_{\varepsilon\overline{S}}, \delta_{\overline{\varepsilon}S}] \varphi^a = -\frac{1}{4} [\delta_K, [\delta_K, [\delta_{\varepsilon\overline{Q}}, \delta_{\overline{\varepsilon}Q}]]] \varphi^a, \quad (4.50)$$

which is the result of the Jacobi identity $\text{JAC}(\delta_K, \delta_{\epsilon\bar{Q}}, \delta_{\bar{\epsilon}Q})$, where

$$\text{JAC}(\delta_1, \delta_2, \delta_3) : [\delta_1, [\delta_2, \delta_3]]\varphi^a + [\delta_2, [\delta_3, \delta_1]]\varphi^a + [\delta_3, [\delta_1, \delta_2]]\varphi^a = 0 , \quad (4.51)$$

the Jacobi identities $\text{JAC}(\delta_K, \delta_{\epsilon\bar{S}}, \delta_{\bar{\epsilon}S})$ and $\text{JAC}(\delta_K, \delta_{\epsilon\bar{S}}, \delta_{\bar{\epsilon}S})$ and the commutation relations

$$[\delta_K, \delta_{\epsilon\bar{S}}]\varphi^a = 0, \quad [\delta_K, \delta_{\bar{\epsilon}S}]\varphi^a = 0 , \quad (4.52)$$

whose validity is easily established. Similarly, we find that $[\delta_K, \delta_{\mathcal{K}^-}]\varphi^a = 0$, since

$$[\delta_K, [\delta_K, [\delta_K, \delta_{\mathcal{P}^-}]]] = 0 , \quad (4.53)$$

follows from (4.49), $\text{JAC}(\delta_K, \delta_{\epsilon\bar{S}}, \delta_{\bar{\epsilon}S})$ and (4.52). The explicit expression for $\delta_{\mathcal{K}^-}\varphi^a$ will not be needed in the remainder of our analysis.

4.5 Dynamical Constraints

By definition, the dynamical constraints are the commutation relations of the dynamical transformations from the following complete set,

$$\delta_{\epsilon\bar{Q}}, \quad \delta_{\bar{\epsilon}Q}, \quad \delta_{\epsilon\bar{S}}, \quad \delta_{\bar{\epsilon}S}, \quad \delta_{\mathcal{P}^-}, \quad \delta_{\mathcal{J}^-}, \quad \delta_{\mathcal{K}^-} . \quad (4.54)$$

We find that (4.40) together with

$$\begin{aligned} [\delta_{\epsilon\bar{Q}}, \delta_{\epsilon'\bar{Q}}]\varphi^a &= 0, & [\delta_{\epsilon\bar{Q}}, \delta_{\epsilon'\bar{S}}]\varphi^a &= 0, & [\delta_{\epsilon\bar{Q}}, \delta_{\bar{\epsilon}S}]\varphi^a &= -i\bar{\epsilon}_m\epsilon^m\delta_{\mathcal{J}^-}\varphi^a, \\ [\delta_{\mathcal{P}^-}, \delta_{\epsilon\bar{Q}}]\varphi^a &= 0, & [\delta_{\mathcal{J}^-}, \delta_{\epsilon\bar{Q}}]\varphi^a &= 0 \end{aligned} \quad (4.55)$$

forms a set of independent dynamical constraints, with the rest of them following upon using the Jacobi identities. The last constraint, $[\delta_{\mathcal{J}^-}, \delta_{\epsilon\bar{Q}}]\varphi^a = 0$, can equivalently be replaced by

$$[\delta_{\mathcal{P}^-}, \delta_{\epsilon\bar{S}}]\varphi^a = 0. \quad (4.56)$$

The dynamical bosonic constraint

$$[\delta_{\mathcal{P}^-}, \delta_{\mathcal{J}^-}]\varphi^a = 0 , \quad (4.57)$$

follows from (4.55); it plays a central role, since all other bosonic dynamical constraints,

$$[\delta_{\mathcal{P}^-}, \delta_{\mathcal{K}^-}]\varphi^a = 0, \quad [\delta_{\mathcal{J}^-}, \delta_{\mathcal{K}^-}]\varphi^a = 0 , \quad (4.58)$$

are derived from it by commuting with δ_K and using $\text{JAC}(\delta_K, \delta_{\mathcal{P}^-}, \delta_{\mathcal{J}^-})$ and $\text{JAC}(\delta_K, \delta_{\mathcal{P}^-}, \delta_{\mathcal{K}^-})$. We will use it to further restrict the form of the supersymmetry transformations.

4.6 Superspace BLG Theory

In the $d = 4$, $N = 4$ SuperYang-Mills case, the dynamical supersymmetry transformations were fixed *uniquely* [12] by solving the constraint (4.57). In the case at hand, this constraint will give us the BLG solution, although not quite uniquely.

The calculation of $\delta_{\mathcal{J}-}\varphi^a$ and then $[\delta_{\mathcal{P}-}, \delta_{\mathcal{J}-}]\varphi^a$ is quite lengthy. (See Appendices C and D.) Here we simply state the result for the odd case

$$[\delta_{\mathcal{P}-}^{\text{odd}}, \delta_{\mathcal{J}-}^{\text{odd}}]\varphi^a = -\frac{1}{4}\mathcal{S}\left(\mathcal{F}\mathcal{O}_1^a + \mathcal{G}\mathcal{O}_2^a\right) + O(f^2) \quad (4.59)$$

where \mathcal{S} , \mathcal{F} and \mathcal{G} are ∂^+ -insertion operators, with \mathcal{S} given in (4.29) and

$$\mathcal{F} = B^* \widehat{\mathcal{U}}_1 + M^* \widehat{\mathcal{U}}_2, \quad \mathcal{G} = C^* \widehat{\mathcal{U}}_3 - D^* \widehat{\mathcal{U}}_4. \quad (4.60)$$

The coefficients

$$\begin{aligned} B^* &\equiv B_\alpha + \alpha - \frac{5}{2}, & M^* &\equiv M_\alpha - C_\alpha - D_\alpha + 3, \\ C^* &\equiv C_\alpha - \alpha - 2, & D^* &\equiv D_\alpha - \alpha - 2 \end{aligned} \quad (4.61)$$

are α -independent thanks to the recursion relation (4.37). This allowed us to pull them outside the sums in ⁵

$$\begin{aligned} \mathcal{O}_1^a &\equiv \frac{\partial^2}{\partial r \partial r'} \left\{ + \sum_{\text{odd}} \left(K_\alpha^{a[r+r',1]} - K_{\alpha+1}^{a[1,r+r']} \right) + 2 \sum_{\text{even}} K_{\alpha+\frac{1}{2}}^{a[r,r']} \right\} \Big|_{\eta=\zeta=r=r'=0} \\ \mathcal{O}_2^a &\equiv \frac{\partial^2}{\partial r \partial r'} \left\{ - \sum_{\text{even}} \left(K_{\alpha+\frac{1}{2}}^{a[r+r',1]} - K_{\alpha+\frac{3}{2}}^{a[1,r+r']} \right) + 2 \sum_{\text{odd}} K_{\alpha+1}^{a[r,r']} \right\} \Big|_{\eta=\zeta=r=r'=0}, \end{aligned} \quad (4.62)$$

where the transverse derivatives ∂ appear via the pairwise insertions of E_r and $E_{r'}$. After performing the differentiations with respect to the parameters r , r' , η and ζ , and setting them to zero, we find that (4.59) is a sum of terms with four \bar{d} 's and two ∂ 's distributed in all possible ways among the three superfields.

The corresponding result in the even case is obtained under the substitution (4.46).

We found two ways in which the commutator (4.59) can vanish.

- The first one is manifest: choose the values of the exponents so that $\mathcal{F} = \mathcal{G} = 0$, that is

$$B^* = M^* = C^* = D^* = 0. \quad (4.63)$$

Noting the dimensional constraint (4.25), this corresponds to

$$A_{-\frac{1}{2}} = 2, \quad B_{-\frac{1}{2}} = 3, \quad M_{-\frac{1}{2}} = 0, \quad C_{-\frac{1}{2}} = D_{-\frac{1}{2}} = \frac{3}{2}. \quad (4.64)$$

⁵The sums in (4.62) contain \mathcal{K}_α^a with $\alpha = 3/2$ and $5/2$, which are outside the range of α for which \mathcal{K}_α^a were originally introduced in (4.31) and (4.34). These \mathcal{K}_α^a are *defined* by the recursion relation (4.36).

As $C_\alpha = D_\alpha$, this imposes $[cd]$ antisymmetry on the structure constants, $f^a{}_{bcd} = -f^a{}_{bdc}$, since the symmetric part drops out in (4.34). In the even case, (4.63) also corresponds to a solution with

$$A_{-1} = \frac{5}{2}, \quad B_{-1} = \frac{7}{2}, \quad M_{-1} = -1, \quad C_{-1} = D_{-1} = 1. \quad (4.65)$$

This time $C_\alpha = D_\alpha$ implies $f^a{}_{bcd} = +f^a{}_{bdc}$, as the antisymmetric part drops out in (4.31). Notice, however, that these solutions have *fractional* powers of ∂^+ ! The fractional solutions have been reported earlier by one of us [14]. We do not know if it is possible to make sense of such solutions. If they survive all the dynamical constraints (4.55) at $O(f)$, which we have not verified, one would have to go to $O(f^2)$ and see if it is still possible to satisfy all the constraints. Even if these solutions lead to algebraically consistent theories, their covariant formulations would likely contain square roots of invariant operators, such as $\sqrt{\partial_\mu \partial^\mu}$, and lead to non-locality. In this paper, we do not consider this type of solution any further.

- If we allow only *integer* values of the exponents, then we find (see Appendix E) that the *only* way to make (4.59) vanish is to choose

$$B^* = 0, \quad M^* = -1, \quad C^* = D^* = -\frac{1}{2}, \quad (4.66)$$

corresponding to

$$A_{-\frac{1}{2}} = B_{-\frac{1}{2}} = 3, \quad M_{-\frac{1}{2}} = -2, \quad C_{-\frac{1}{2}} = D_{-\frac{1}{2}} = 1, \quad (4.67)$$

and require total antisymmetry of $f^a{}_{bcd}$ under the interchange of the last *three* indices,

$$f^a{}_{bcd} = f^a{}_{[bcd]}. \quad (4.68)$$

We found this by considering a particular subset of terms in (4.59) with all four \bar{d} 's and both ∂ 's acting on the same superfield. Under the above conditions, and *only* in this case, the net contribution of these terms vanishes. The vanishing of the other subsets follow from the kinematical supersymmetry and the other linear symmetry transformations. We find that (4.67) together with (4.68) correspond to the covariantly formulated BLG theory, which is known to be algebraically consistent.

Indeed, with the values of the exponents in (4.67) and using the antisymmetry property (4.68),

we find (see Appendix G) that (4.39) reduces to ⁶

$$\delta_{\bar{\epsilon}Q}^{int} \varphi^a = -8 \bar{\epsilon}_m f^a_{bcd} \frac{1}{\partial^+} \left(\partial^+ \varphi^b \cdot \frac{1}{\partial^+} \left((\sqrt{2} \partial^+ \varphi^c) (\partial^+ d^m \bar{\varphi}^d) - i (\partial^+ \bar{d}_n \varphi^c) (d^{mn} \bar{\varphi}^d) \right) \right), \quad (4.69)$$

where four \bar{d} 's are absorbed into the conjugated superfield $\bar{\varphi}^d$. Note that the two terms are required by chirality. After a rescaling of f^a_{bcd} , this matches the corresponding expression in [4] derived by direct light-cone gauge fixing of the BLG theory. The expression for $\delta_{\epsilon\bar{Q}}^{int} \varphi^a$ following from (4.34) is much more complicated. However, the inverse of (4.3),

$$\delta_{\epsilon\bar{Q}}^{int} \varphi^a = \frac{\bar{d}^{[4]}}{2\partial^{+2}} (\delta_{\bar{\epsilon}Q}^{int} \varphi^a)^*, \quad (4.70)$$

provides an alternative (and compact) expression for it.

As we show in Appendix F, there is no such integer solution in the even case.

To summarize, we found that the odd Ansatz (4.34) yields the BLG theory for the interaction part of the dynamical supersymmetry with the values of the exponents given in (4.67) and the coefficients f^a_{bcd} satisfying the antisymmetry condition (4.68).

The basic result of this paper is that this is *the only* solution to the constraints of the $OSp(2, 2|8)$ superalgebra if we allow only integer powers of ∂^+ .

Having matched the solution (4.67) with the BLG theory, we have found that all the dynamical constraints (4.55) are satisfied at $O(f)$ thanks to the antisymmetry of f^a_{bcd} , and at $O(f^2)$ thanks to the Fundamental Identity [1, 2, 15]

$$f^a_{bc[d} f^b_{efg]} = 0, \quad (4.71)$$

which identifies f^a_{bcd} with the structure constants of a 3-Lie algebra. Note that this symmetry is a global one in our formalism. There is no gauge field in the algebra. In the light-cone formulation this follows since the gauge field can be completely integrated out after the gauge fixing [3]. Note also that at this level we have not obtained any quantization constraint on the structure constant. We expect that to happen when we further analyse the quantum properties of the theory.

The knowledge of the dynamical supersymmetry transformations fixes the theory uniquely, with all other dynamical transformations following by commutations. In particular, calculating the Hamiltonian shift $\delta_{P-\varphi^a}$ using (4.40) determines the full interacting equations of motion,

$$\partial^- \varphi^a = i \delta_{P-\varphi^a}. \quad (4.72)$$

⁶In the $d = 4$ SuperYang-Mills case, even after choosing the light-cone gauge, there remains a residual gauge symmetry on the transverse vector fields with gauge parameter satisfying $\partial^+ \Lambda = 0$ [4]. As a result, the interacting supersymmetry transformations are obtained by covariantizing the transverse derivative [12]. In the $d = 3$ SuperChern-Simons case (the BLG theory), there is no such residual symmetry (as the transverse vector fields are not independent degrees of freedom in the light-cone gauge [3]), and we are unable to write the interacting transformations by generalizing the transverse derivative in (3.9).

5 BLG Hamiltonian as a Quadratic Form

The full dynamical supersymmetry transformations in the light-cone superspace formulation of the BLG theory are given by the sum of the free transformations (3.9),

$$\delta_{\bar{\epsilon}\mathcal{Q}}^{free} \varphi^a = \frac{1}{\sqrt{2}} \bar{\epsilon}_m q^m \frac{\partial}{\partial^+} \varphi^a, \quad \delta_{\epsilon\bar{\mathcal{Q}}}^{int} \varphi^a = \frac{1}{\sqrt{2}} \epsilon^m \bar{q}_m \frac{\partial}{\partial^+} \varphi^a \quad (5.1)$$

and the interaction parts (4.69) and (4.70). Using (4.40), we can now find the complete BLG Hamiltonian shift $\delta_{\mathcal{P}^-} \varphi^a$. Its free part is given by the standard expression (3.11)

$$\delta_{\mathcal{P}^-}^{free} \varphi^a = -\frac{i}{2} \frac{\partial^2}{\partial^+} \varphi^a. \quad (5.2)$$

To write the corresponding light-cone superspace Hamiltonian H , we need to introduce a metric $h_{ab} = h_{ba}$ for the gauge indices. Then the free Hamiltonian is

$$H^{free} = h_{ab} \int dz \bar{\varphi}^a \frac{\partial^2}{\partial^{+2}} \varphi^b, \quad (5.3)$$

where $dz = d^3x d^4\theta d^4\bar{\theta}$, and it is related to the Hamiltonian shift via the functional derivative,

$$\frac{\delta H^{free}}{\delta \varphi^a} = 8i h_{ab} \partial^+ (\delta_{\mathcal{P}^-}^{free} \varphi^b), \quad (5.4)$$

as can be easily verified using the basic rule of functional differentiation [8]

$$\frac{\delta \varphi^a(z)}{\delta \varphi^b(z')} = d^{[4]} \delta(z - z') \delta_b^a \quad (5.5)$$

and the inside-out constraint. The full Hamiltonian H can then be found by integrating (5.4) with the full Hamiltonian shift $\delta_{\mathcal{P}^-} \varphi^a$. However, instead of doing the complicated integration, we can start with a natural guess for H and verify that it yields the correct $\delta_{\mathcal{P}^-} \varphi^a$ upon differentiation. Such a guess is provided by the quadratic form property of the light-cone superspace Hamiltonian in maximally supersymmetric theories, discovered in [12]. If this property holds in the BLG theory, then we should have

$$\begin{aligned} H &= \frac{i}{\sqrt{2}} h_{ab} \int dz \bar{\mathcal{Q}}_m^a \frac{1}{\partial^+} \mathcal{Q}^{bm} \equiv \frac{1}{\sqrt{2}} \langle \mathcal{Q}, \mathcal{Q} \rangle \\ &= \frac{i}{2\sqrt{2}} h_{ab} \int dz \left(\bar{q}_m \frac{\partial}{\partial^+} \bar{\varphi}^a + \bar{\mathcal{W}}_m^a \right) \frac{1}{\partial^+} \left(q^m \frac{\partial}{\partial^+} \varphi^b + \mathcal{W}^{bm} \right), \end{aligned} \quad (5.6)$$

where the hermitian form $\langle \cdot, \cdot \rangle$ is defined in (A.3). The \mathcal{Q}^{am} and \mathcal{W}^{am} are defined by

$$\delta_{\bar{\epsilon}\mathcal{Q}} \varphi^a \equiv \bar{\epsilon}_m \mathcal{Q}^{am}, \quad \delta_{\epsilon\bar{\mathcal{Q}}}^{int} \varphi^a \equiv \frac{1}{\sqrt{2}} \bar{\epsilon}_m \mathcal{W}^{am} \quad (5.7)$$

together with $\overline{\mathcal{Q}}_m^a \equiv (\mathcal{Q}^{am})^*$ and $\overline{\mathcal{W}}_m^a \equiv (\mathcal{W}^{am})^*$. From (4.69), we have, explicitly,

$$\begin{aligned}\mathcal{W}^{am} &= 8f_{bcd}^a \frac{1}{\partial^+} \left(\partial^+ \varphi^b \cdot \frac{1}{\partial^+} \left(2\partial^+ d^m \overline{\varphi}^c \cdot \partial^+ \varphi^d - i\sqrt{2} d^{mn} \overline{\varphi}^c \cdot \partial^+ \overline{d}_n \varphi^d \right) \right), \\ \overline{\mathcal{W}}_m^a &= 8f_{bcd}^a \frac{1}{\partial^+} \left(\partial^+ \overline{\varphi}^b \cdot \frac{1}{\partial^+} \left(2\partial^+ \overline{d}_m \varphi^c \cdot \partial^+ \overline{\varphi}^d - i\sqrt{2} \overline{d}_{mn} \varphi^c \cdot \partial^+ d^n \overline{\varphi}^d \right) \right).\end{aligned}\quad (5.8)$$

At the free level, after several integrations by parts, the use of the inside-out constraint (4.3) and the anticommutator (3.2), we find that (5.6) reproduces H^{free} in (5.3). For the $O(f)$ part of H , we have

$$H^{(1)} = \frac{i}{2\sqrt{2}} h_{ab} \int dz \left(q^m \frac{\partial}{\partial^+} \overline{\varphi}^a \cdot \frac{1}{\partial^+} \overline{\mathcal{W}}_m^b \right) + c.c. \quad (5.9)$$

whereas the $O(f^2)$ part is

$$H^{(2)} = \frac{i}{2\sqrt{2}} h_{ab} \int dz \left(\overline{\mathcal{W}}_m^a \frac{1}{\partial^+} \mathcal{W}^{bm} \right) \quad (5.10)$$

We have *not* verified that functional differentiation of $H^{(1)}$ and $H^{(2)}$ reproduces $\delta_{\mathcal{P}^-}^{(1)} \varphi^a$ and $\delta_{\mathcal{P}^-}^{(2)} \varphi^a$ as follow from (4.40). By analogy with the $d = 4$ SuperYang-Mills [12], we expect that such a verification at $O(f)$ would require *total* antisymmetry of the structure constants,

$$f_{abcd} \equiv h_{ag} f^g_{bcd} = f_{[abcd]} \quad (5.11)$$

whereas at $O(f^2)$ the Fundamental Identity (4.71) would be required. What we *have* verified, is that when $H^{(1)}$ is written in terms of the component fields A^a , C^{mna} , χ^{ma} (and their conjugates), its “ C -only” part, after using (5.11), is

$$H^{(1)}|_{C\text{-only}} = -16 f_{abcd} \int d^3x (\overline{C}_{ij}^a \partial C^{ijb}) \frac{1}{\partial^+} (\overline{C}_{mn}^c \partial^+ C^{mnd}) \quad (5.12)$$

which matches the corresponding part in the light-cone BLG Hamiltonian [3]. This is enough to show that (5.9) can be transformed to the form proposed by Nilsson [3],

$$H^{(1)} = -8 f_{abcd} \int dz (\varphi^a \partial \varphi^b) \frac{1}{\partial^+} (\overline{\varphi}^c \partial^+ \overline{\varphi}^d) + c.c. \quad (5.13)$$

as the two expressions match on the level of “ C -only” terms. (This way of verifying equivalence of two different superfield expressions was also used in [12].) As the calculations involved are quite nontrivial (see Appendix H), we feel that this provides sufficient evidence for the correctness of the full Hamiltonian given as the quadratic form (5.6).

6 Conclusion and Discussion

In this paper, we have constructed the superconformal theory of Bagger, Lambert and Gustavsson in three dimensions by requiring closure of $OSp(2, 2|8)$ on constrained chiral superfields in light-cone superspace. The algebra splits into kinematical and dynamical operators: kinematical operators act linearly on the superfields, while dynamical operators contain terms linear (free theory) and non-linear (interactions) in the superfields.

A feature of any superconformal theory is that all dynamics is algebraically determined by its supersymmetry transformations. We first determined ansätze for the dynamical supersymmetry transformations which satisfied all kinematical constraints. These constraints required that the theory contain a fourth order tensor f^a_{bcd} . By demanding commutation of the transformations generated by the Hamiltonian and boost, we were able to narrow down the form of the supersymmetry transformations to two choices.

One is the BLG theory, which requires the antisymmetry of the $f^a_{[bcd]}$ type.

The other “solutions” to the algebraic constraints entail fractional powers of light-cone derivatives ∂^+ , and partial symmetries whereby f^a_{bcd} are symmetric (antisymmetric) under $c \leftrightarrow d$ for the even (odd) cases. In this paper, we have not checked their consistency with the full algebra, since their covariant formulation would likely lead to square roots of covariant operators such as $\sqrt{\partial_\mu \partial^\mu}$, and therefore non-local interactions.

Our formulation of the BLG theory has many analogies to $N = 4$ SuperYang-Mills, since they use the same chiral superfield. In particular, the light-cone superspace Hamiltonian of both theories can be written as a quadratic form. In the Yang-Mills case we found a tensor of the form f^a_{bc} which satisfied Lie algebra Jacobi identities from closure of the algebra. In the BLG case, we found f^a_{bcd} , which will satisfy the fundamental identity of BLG, from closure as well⁷. Our formalism has invoked $SO(8)$ as the R -symmetry. We can now use this formalism and relax part of the R -symmetry to search for other superconformal theories.

The same chiral superfield in $d = 5$ and $d = 6$ dimensions forms a linear representation of the superconformal group appropriate to these dimensions. We intend to use these algebraic techniques to study their possible interactions in future publications.

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⁷In a similar approach, the authors of [16] considered standard $N = 8$ superspace constraints and also obtained the BLG theory as a special solution described by the same fourth rank tensor $f^a_{[bcd]}$.

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Appendix

A Light-Cone $OSp(2, 2|8)$ Algebra

The free theory (operator) representation of the $OSp(2, 2|8)$ superalgebra used in this paper is given by

$$\begin{aligned}
P^+ &= -i\partial^+, \quad P = -i\partial, \quad \mathcal{P}^- = -\frac{i}{2}\frac{\partial^2}{\partial^+}, \\
J^+ &= ix\partial^+, \quad J^{+-} = i(\mathcal{A} + \frac{x}{2}\partial + \frac{1}{2}), \quad \mathcal{J}^- = -i\frac{\partial}{\partial^+}\mathcal{A}, \\
D &= i(\mathcal{A} - \frac{x}{2}\partial), \quad K^+ = ix^2\partial^+, \quad K = 2ix\mathcal{A}, \quad \mathcal{K}^- = 2i\frac{1}{\partial^+}\mathcal{A}(\mathcal{A} - \frac{1}{2}), \\
T^m{}_n &= \frac{i}{\sqrt{2}\partial^+} \left(q^m \bar{q}_n - \frac{1}{4} \delta^m{}_n q^k \bar{q}_k \right), \quad T = \frac{i}{4\sqrt{2}\partial^+} (q^k \bar{q}_k - \bar{q}_k q^k), \\
T^{mn} &= \frac{i}{\sqrt{2}\partial^+} q^m q^n, \quad \bar{T}_{mn} = \frac{i}{\sqrt{2}\partial^+} \bar{q}_m \bar{q}_n, \\
\mathcal{Q}^m &= \frac{1}{\sqrt{2}} \frac{\partial}{\partial^+} q^m, \quad s^m = ixq^m, \quad \mathcal{S}^m = -iq^m \frac{1}{\partial^+} \mathcal{A}, \\
\bar{\mathcal{Q}}_m &= \frac{1}{\sqrt{2}} \frac{\partial}{\partial^+} \bar{q}_m, \quad \bar{s}_m = -ix\bar{q}_m, \quad \bar{\mathcal{S}}_m = i\bar{q}_m \frac{1}{\partial^+} \mathcal{A}
\end{aligned} \tag{A.1}$$

together with the kinematical supersymmetry generators,

$$\begin{aligned}
q^m &= -\frac{\partial}{\partial \bar{\theta}_m} + \frac{i}{\sqrt{2}} \theta^m \partial^+, \quad \bar{q}_m = \frac{\partial}{\partial \theta^m} - \frac{i}{\sqrt{2}} \bar{\theta}_m \partial^+, \\
\mathcal{A} &= x^- \partial^+ - \frac{x}{2} \partial - \frac{1}{2} \mathcal{N} + \frac{1}{2}, \quad \mathcal{N} = \theta^k \frac{\partial}{\partial \theta^k} + \bar{\theta}_k \frac{\partial}{\partial \bar{\theta}_k}.
\end{aligned} \tag{A.2}$$

The generators of the conformal group are chosen to be *hermitian* with respect to the following hermitian form ⁸

$$\langle \varphi_1, \varphi_2 \rangle = i \int d^3x d^4\theta d^4\bar{\theta} \varphi_1^* \frac{1}{\partial^+} \varphi_2. \quad (\text{A.3})$$

The hermitian conjugate \mathcal{O}^\dagger of an operator \mathcal{O} is defined by

$$\langle \varphi_1, \mathcal{O}\varphi_2 \rangle = \langle \mathcal{O}^\dagger \varphi_1, \varphi_2 \rangle \quad (\text{A.4})$$

so that $\mathcal{O}^\dagger = \mathcal{O}^*$ if \mathcal{O} does not depend on x^- . The dependence on x^- in the above generators comes via the dependence on \mathcal{A} . Direct computation gives

$$\mathcal{A}^\dagger = -\mathcal{A} - \frac{1}{2} \quad (\text{A.5})$$

and then hermiticity properties of all the generators follow,

$$\begin{aligned} \mathcal{O}^\dagger &= +\mathcal{O} & \text{for } \mathcal{O} = (P^+, P, \mathcal{P}^-, J^+, J^{+-}, \mathcal{J}^-, D, K^+, K, \mathcal{K}^-) \\ (\mathcal{O}^m)^\dagger &= -\overline{\mathcal{O}}_m & \text{for } \mathcal{O}^m = (q^m, \mathcal{Q}^m, s^m, \mathcal{S}^m) \\ (T^m{}_n)^\dagger &= T^n{}_m, \quad (T)^\dagger = T, \quad (T^{mn})^\dagger = -\overline{T}_{mn}. \end{aligned} \quad (\text{A.6})$$

Using the following basic commutation properties

$$\begin{aligned} [\partial^+, x^-] &= -1, \quad [\partial, x] = 1, \quad \{q^m, q^n\} = \{\overline{q}_m, \overline{q}_n\} = 0, \\ \{q^m, \overline{q}_n\} &= i\sqrt{2}\delta^m{}_n\partial^+, \quad [\mathcal{A}, q^m] = \frac{1}{2}q^m, \quad [\mathcal{A}, \overline{q}_m] = \frac{1}{2}\overline{q}_m, \\ [\mathcal{A}, \partial^+] &= \partial^+, \quad [\mathcal{A}, \frac{1}{\partial^+}] = -\frac{1}{\partial^+}, \quad [\mathcal{A}, x] = -\frac{1}{2}x, \quad [\mathcal{A}, \partial] = \frac{1}{2}\partial \end{aligned} \quad (\text{A.7})$$

one can verify that the algebra closes. The *non-vanishing* (anti)commutators of $OSp(2, 2|8)$ are as follows.⁹

⁸Complex conjugation, denoted by $*$, interchanges the order of operands: $(\mathcal{O}_1 \dots \mathcal{O}_n)^* = \mathcal{O}_n^* \dots \mathcal{O}_1^*$ irrespective of whether \mathcal{O} 's are bosonic or fermionic objects. See Appendix A in [4] for more details. However, our rule for conjugation of *fermionic parameters* differs from [4]: we define $(\varepsilon^m)^* = -\overline{\varepsilon}_m$ so that $(\varepsilon^m \overline{q}_m \varphi)^* = +\overline{\varepsilon}_m q^m \overline{\varphi}$.

⁹We note that P 's commute with P 's, K 's commute with K 's, D commutes with J 's, and R -symmetry generators T 's commute with all other bosonic generators.

- In the $Sp(2, 2) \sim SO(3, 2)$ conformal group sector:

$$\begin{aligned}
[J^{+-}, J^+] &= iJ^+, \quad [J^{+-}, \mathcal{J}^-] = -i\mathcal{J}^-, \quad [J^+, \mathcal{J}^-] = iJ^{+-}. \\
[J^{+-}, P^+] &= iP^+, \quad [J^{+-}, \mathcal{P}^-] = -i\mathcal{P}^-, \quad [J^{+-}, K^+] = iK^+, \quad [J^{+-}, \mathcal{K}^-] = -i\mathcal{K}^-. \\
[J^+, P] &= -iP^+, \quad [J^+, \mathcal{P}^-] = -iP, \quad [J^+, K] = -iK^+, \quad [J^+, \mathcal{K}^-] = -iK. \\
[J^-, P^+] &= -iP, \quad [J^-, P] = -i\mathcal{P}^-, \quad [J^-, K^+] = -iK, \quad [J^-, K] = -iK^-. \\
[K^+, P] &= 2iJ^+, \quad [K^+, \mathcal{P}^-] = 2i(J^{+-} - D). \\
[\mathcal{K}^-, P] &= 2i\mathcal{J}^-, \quad [\mathcal{K}^-, P^+] = -2i(J^{+-} + D). \\
[K, P^+] &= -2iJ^+, \quad [K, P^-] = -2iJ^-, \quad [K, P] = 2iD. \\
[D, P^+] &= iP^+, \quad [D, P] = iP, \quad [D, \mathcal{P}^-] = i\mathcal{P}^-. \\
[D, K^+] &= -iK^+, \quad [D, K] = -iK, \quad [D, \mathcal{K}^-] = -i\mathcal{K}^-. \tag{A.8}
\end{aligned}$$

- In the $SO(8)$ R -symmetry group sector:

$$\begin{aligned}
[T^m{}_n, T^k{}_l] &= \delta^m{}_l T^k{}_n - \delta^k{}_n T^m{}_l, \\
[T^m{}_n, T^{kl}] &= \frac{1}{2} \delta^m{}_n T^{kl} - \delta^k{}_n T^{ml} + \delta^l{}_n T^{mk}, \quad [T, T^{mn}] = -T^{mn}. \\
[T^n{}_m, \bar{T}_{kl}] &= \frac{1}{2} \delta^n{}_m \bar{T}_{kl} - \delta^n{}_k \bar{T}_{ml} + \delta^n{}_l \bar{T}_{mk}, \quad [T, \bar{T}_{mn}] = -\bar{T}_{mn}. \\
[T^{mn}, \bar{T}_{kl}] &= \delta^m{}_k T^n{}_l - \delta^m{}_l T^n{}_k + \delta^n{}_l T^m{}_k - \delta^n{}_k T^m{}_l + (\delta^m{}_k \delta^n{}_l - \delta^m{}_l \delta^n{}_k) T. \tag{A.9}
\end{aligned}$$

- R -symmetry group action on the fermionic generators:

$$\begin{aligned}
[T^m{}_n, q^k] &= \frac{1}{4} \delta^m{}_n q^k - \delta^k{}_n q^m, \quad [T, q^m] = -\frac{1}{2} q^m, \quad [\bar{T}_{mn}, q^k] = \delta^k{}_m \bar{q}_n - \delta^k{}_n \bar{q}_m, \\
[T^n{}_m, \bar{q}_k] &= -\frac{1}{4} \delta^n{}_m \bar{q}_k + \delta^n{}_k \bar{q}_m, \quad [T, \bar{q}_m] = \frac{1}{2} \bar{q}_m, \quad [T^{mn}, \bar{q}_k] = \delta^m{}_k q^n - \delta^n{}_k q^m \tag{A.10}
\end{aligned}$$

and identically for s^m and \bar{s}_m , \mathcal{Q}^m and $\bar{\mathcal{Q}}_m$, \mathcal{S}^m and $\bar{\mathcal{S}}_m$.

- Conformal group action on the fermionic generators:

$$\begin{aligned}
[J^{+-}, q^m] &= \frac{i}{2} q^m & [J^{+-}, \bar{q}_m] &= \frac{i}{2} \bar{q}_m \\
[J^{+-}, \mathcal{Q}^m] &= -\frac{i}{2} \mathcal{Q}^m & [J^{+-}, \bar{\mathcal{Q}}_m] &= -\frac{i}{2} \bar{\mathcal{Q}}_m \\
[J^{+-}, s^m] &= \frac{i}{2} s^m & [J^{+-}, \bar{s}_m] &= \frac{i}{2} \bar{s}_m \\
[J^{+-}, \mathcal{S}^m] &= -\frac{i}{2} \mathcal{S}^m & [J^{+-}, \bar{\mathcal{S}}_m] &= -\frac{i}{2} \bar{\mathcal{S}}_m \\
[J^+, \mathcal{Q}^m] &= -\frac{i}{\sqrt{2}} q^m & [J^+, \bar{\mathcal{Q}}_m] &= -\frac{i}{\sqrt{2}} \bar{q}_m \\
[J^+, \mathcal{S}^m] &= \frac{i}{2} s^m & [J^+, \bar{\mathcal{S}}_m] &= \frac{i}{2} \bar{s}_m \\
[\mathcal{J}^-, q^m] &= -\frac{i}{\sqrt{2}} \mathcal{Q}^m & [\mathcal{J}^-, \bar{q}_m] &= -\frac{i}{\sqrt{2}} \bar{\mathcal{Q}}_m \\
[\mathcal{J}^-, s^m] &= i \mathcal{S}^m & [\mathcal{J}^-, \bar{s}_m] &= i \bar{\mathcal{S}}_m \\
[P^+, \mathcal{S}^m] &= q^m & [P^+, \bar{\mathcal{S}}_m] &= -\bar{q}_m \\
[P, s^m] &= q^m & [P, \bar{s}_m] &= -\bar{q}_m \\
[P, \mathcal{S}^m] &= \frac{1}{\sqrt{2}} \mathcal{Q}^m & [P, \bar{\mathcal{S}}_m] &= -\frac{1}{\sqrt{2}} \bar{\mathcal{Q}}_m \\
[\mathcal{P}^-, s^m] &= \sqrt{2} \mathcal{Q}^m & [\mathcal{P}^-, \bar{s}_m] &= -\sqrt{2} \bar{\mathcal{Q}}_m \\
[K^+, \mathcal{Q}^m] &= -\sqrt{2} s^m & [K^+, \bar{\mathcal{Q}}_m] &= \sqrt{2} \bar{s}_m \\
[K, q^m] &= s^m & [K, \bar{q}_m] &= -\bar{s}_m \\
[K, \mathcal{Q}^m] &= \sqrt{2} \mathcal{S}^m & [K, \bar{\mathcal{Q}}_m] &= -\sqrt{2} \bar{\mathcal{S}}_m \\
[\mathcal{K}^-, q^m] &= -2 \mathcal{S}^m & [\mathcal{K}^-, \bar{q}_m] &= 2 \bar{\mathcal{S}}_m \\
[D, q^m] &= \frac{i}{2} q^m & [D, \bar{q}_m] &= \frac{i}{2} \bar{q}_m \\
[D, \mathcal{Q}^m] &= \frac{i}{2} \mathcal{Q}^m & [D, \bar{\mathcal{Q}}_m] &= \frac{i}{2} \bar{\mathcal{Q}}_m \\
[D, s^m] &= -\frac{i}{2} s^m & [D, \bar{s}_m] &= -\frac{i}{2} \bar{s}_m \\
[D, \mathcal{S}^m] &= -\frac{i}{2} \mathcal{S}^m & [D, \bar{\mathcal{S}}_m] &= -\frac{i}{2} \bar{\mathcal{S}}_m. \tag{A.11}
\end{aligned}$$

- Anticommutation relations:

$$\begin{aligned}
\{q^m, \bar{q}_n\} &= -\sqrt{2}\delta^m{}_n P^+ & \{\mathcal{Q}^m, \bar{\mathcal{Q}}_n\} &= -\sqrt{2}\delta^m{}_n \mathcal{P}^- \\
\{s^m, \bar{s}_n\} &= \sqrt{2}\delta^m{}_n K^+ & \{\mathcal{S}^m, \bar{\mathcal{S}}_n\} &= \frac{1}{\sqrt{2}}\delta^m{}_n \mathcal{K}^- \\
\{q^m, \bar{\mathcal{Q}}_n\} &= -\delta^m{}_n P & \{\bar{q}_m, \mathcal{Q}^n\} &= -\delta^n{}_m P \\
\{q^m, \bar{s}_n\} &= -i\sqrt{2}\delta^m{}_n J^+ & \{\bar{q}_m, s^n\} &= i\sqrt{2}\delta^n{}_m J^+ \\
\{\mathcal{Q}^m, \bar{\mathcal{S}}_n\} &= -i\delta^m{}_n \mathcal{J}^- & \{\bar{\mathcal{Q}}_m, \mathcal{S}^n\} &= i\delta^n{}_m \mathcal{J}^- \\
\{q^m, \mathcal{S}^n\} &= \frac{1}{\sqrt{2}}T^{mn} & \{\bar{q}_m, \bar{\mathcal{S}}_n\} &= -\frac{1}{\sqrt{2}}T_{mn} \\
\{\mathcal{Q}^m, s^n\} &= T^{mn} & \{\bar{\mathcal{Q}}_m, \bar{s}_n\} &= -\bar{T}_{mn} \\
\{q^m, \bar{\mathcal{S}}_n\} &= \frac{i}{\sqrt{2}}(J^{+-} + D)\delta^m{}_n - \frac{1}{\sqrt{2}}(T^m{}_n + \frac{1}{2}T\delta^m{}_n) \\
\{\bar{q}_m, \mathcal{S}^n\} &= -\frac{i}{\sqrt{2}}(J^{+-} + D)\delta^n{}_m - \frac{1}{\sqrt{2}}(T^n{}_m + \frac{1}{2}T\delta^n{}_m) \\
\{\mathcal{Q}^m, \bar{s}_n\} &= -i(J^{+-} - D)\delta^m{}_n - (T^m{}_n + \frac{1}{2}T\delta^m{}_n) \\
\{\bar{\mathcal{Q}}_m, s^n\} &= i(J^{+-} - D)\delta^n{}_m - (T^n{}_m + \frac{1}{2}T\delta^n{}_m). \tag{A.12}
\end{aligned}$$

This set of commutation relations is invariant under hermitian conjugation (A.6). When using these *operator* commutation relations to write the corresponding ones for *transformations*, one has to note the following minus sign

$$[\delta_{\mathcal{O}_1}, \delta_{\mathcal{O}_2}]\varphi^a = -[\mathcal{O}_1, \mathcal{O}_2]\varphi^a, \tag{A.13}$$

where $\delta_{\mathcal{O}}\varphi^a = \mathcal{O}\varphi^a$ are *free* theory transformations. The resulting set of commutation relations is required to be satisfied by the *interacting* theory transformations as well.

B Useful Identities

We present a set of useful formulae and identities:

- Commutators:

$$[E_\varepsilon, \theta^m \bar{q}_n] = \varepsilon^m \frac{\partial}{\partial \varepsilon^n} E_\varepsilon, \quad [E_\eta, \theta^m q_n] = \eta^m \left(\frac{\partial}{\partial \eta^n} - i\sqrt{2}\bar{\theta}_n \right) E_\eta. \tag{B.1}$$

$$[\mathcal{A}, \bar{d}_m] = \frac{1}{2} \bar{d}_m, \quad [E_\varepsilon E_\eta \partial^{+k}, \mathcal{A}] = \left(\frac{1}{2} \varepsilon \frac{\partial}{\partial \varepsilon} + \frac{1}{2} \eta \frac{\partial}{\partial \eta} - k \right) E_\varepsilon E_\eta \partial^{+k}. \quad (\text{B.2})$$

$$[E_\eta, \theta^m] = \hat{\eta}^m E_\eta, \quad [E_\eta, \theta^m \theta^n \partial^+] = (\theta^m \eta^n - \theta^n \eta^m + \eta^m \hat{\eta}^n) E_\eta. \quad (\text{B.3})$$

- The Master Formula:

Consider the commutator of a transformation linear in φ^a , $\delta_{\mathcal{O}} \varphi^a = \mathcal{O} \varphi^a$, with a transformation nonlinear in φ 's,

$$\delta_X \varphi^a \equiv f^a_{bcd} ((X_1 \varphi^b) X_2 ((X_3 \varphi^c) (X_4 \varphi^d))), \quad (\text{B.4})$$

where X_i are operators. In terms of the insertion operators, their commutator can be written as the master formula

$$[\delta_{\mathcal{O}}, \delta_X] \varphi^a = \left(\sum_{i=1}^4 [X_i, \mathcal{O}] X_i^{-1} \mathcal{U}_i + \{\mathcal{O}\}_{12} + \{\mathcal{O}\}_{34} \right) \delta_X \varphi^a, \quad (\text{B.5})$$

where

$$\{\mathcal{O}\}_{12} \equiv \mathcal{O} \mathcal{U}_1 + \mathcal{O} \mathcal{U}_2 - \mathcal{O}, \quad \{\mathcal{O}\}_{34} \equiv \mathcal{O} \mathcal{U}_3 + \mathcal{O} \mathcal{U}_4 - \mathcal{U}_2 \mathcal{O}, \quad (\text{B.6})$$

account for the deviation from Leibnitz's rule. Indeed if \mathcal{D} is a derivative operator, then $\{\mathcal{D}\}_{12} = \{\mathcal{D}\}_{34} = 0$. Given two *derivative* operators \mathcal{D} and \mathcal{D}' , commuting with ∂^+ , we find that (even if \mathcal{D} and \mathcal{D}' do not commute),

$$\left\{ \frac{\mathcal{D} \mathcal{D}'}{\partial^+} \right\}_{ij} = \frac{\partial}{\partial r} \frac{\partial}{\partial r'} \frac{1}{\partial^+} \left((\partial^+ \mathcal{E}_r \mathcal{E}_{r'} \mathcal{U}_i) (\partial^+ \mathcal{E}_{-r} \mathcal{E}_{-r'} \mathcal{U}_j) \right) \Big|_{r=r'=0}, \quad (\text{B.7})$$

for $(ij) = (12), (34)$, and where

$$\mathcal{E}_r \equiv e^{r \hat{\mathcal{D}}}, \quad \mathcal{E}_{r'} \equiv e^{r' \hat{\mathcal{D}}'}. \quad (\text{B.8})$$

A frequently used identity is

$$\hat{\mathcal{O}} \mathcal{U}_i - \hat{\mathcal{O}} \mathcal{U}_j = \frac{\partial}{\partial r} \left((e^{r \hat{\mathcal{O}}} \mathcal{U}_i) (e^{-r \hat{\mathcal{O}}} \mathcal{U}_j) \right) \Big|_{r=0}. \quad (\text{B.9})$$

Noting that $\hat{\bar{q}}_m = \hat{\bar{d}}_m - i\sqrt{2} \bar{\theta}_m$ and $\hat{q}^m = \hat{d}^m + i\sqrt{2} \theta^m$, we have

$$(e^{\varepsilon \hat{\bar{q}}} \mathcal{U}_i) (e^{-\varepsilon \hat{\bar{q}}} \mathcal{U}_j) = (e^{\varepsilon \hat{\bar{d}}} \mathcal{U}_i) (e^{-\varepsilon \hat{\bar{d}}} \mathcal{U}_j), \quad (e^{\bar{\varepsilon} \hat{q}} \mathcal{U}_i) (e^{-\bar{\varepsilon} \hat{q}} \mathcal{U}_j) = (e^{\bar{\varepsilon} \hat{d}} \mathcal{U}_i) (e^{-\bar{\varepsilon} \hat{d}} \mathcal{U}_j). \quad (\text{B.10})$$

Using then $\{d^m, \bar{d}_n\} = -i\sqrt{2}\delta^m{}_n\partial^+$, and $d^m\varphi^a = 0$, we find that ¹⁰

$$\mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)} = \epsilon^m \frac{\partial}{\partial \eta^m} \mathcal{K}_\alpha^{a(0, \eta, \zeta)}, \quad \mathcal{K}_\alpha^{a(\bar{\varepsilon}, \eta, \zeta)} = i\sqrt{2}\bar{\varepsilon}_m \eta^m \mathcal{S} \mathcal{K}_\alpha^{a(0, \eta, \zeta)}. \quad (\text{B.11})$$

- Selected Applications

The master formula (B.5) and use of (B.1) yields,

$$\begin{aligned} [\delta_{SU(4)}, \delta_{\varepsilon \bar{\mathcal{Q}}}^{int}] \varphi^a &= -\omega^m{}_n \frac{1}{\sqrt{2}} \sum \left(\varepsilon^n \frac{\partial}{\partial \varepsilon^m} + \eta^n \frac{\partial}{\partial \eta^m} + \zeta^n \frac{\partial}{\partial \zeta^m} \right) \mathcal{K}_\alpha^{a(\epsilon, \eta, \zeta)}, \\ [\delta_{U(1)}, \delta_{\varepsilon \bar{\mathcal{Q}}}^{int}] \varphi^a &= -\omega \frac{1}{\sqrt{2}} \sum \left(\varepsilon \frac{\partial}{\partial \varepsilon} + \eta \frac{\partial}{\partial \eta} + \zeta \frac{\partial}{\partial \zeta} \right) \mathcal{K}_\alpha^{a(\epsilon, \eta, \zeta)}. \end{aligned} \quad (\text{B.12})$$

Since \mathcal{A} is not a derivative operator, the master formula (B.5) gets a contribution from the “triplets” (B.6), and using (B.2), we obtain

$$\begin{aligned} [\delta_{\mathcal{A}}, \delta_{\varepsilon \bar{\mathcal{Q}}}^{int}] \varphi^a &= \frac{1}{\sqrt{2}} \sum \left[1 + \frac{1}{2} \left(\epsilon \frac{\partial}{\partial \epsilon} + \eta \frac{\partial}{\partial \eta} + \zeta \frac{\partial}{\partial \zeta} \right) \right. \\ &\quad \left. + (A_\alpha - B_\alpha + M_\alpha - C_\alpha - D_\alpha) \right] \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)}, \end{aligned} \quad (\text{B.13})$$

which leads to the dimensional constraint (4.25).

We also get

$$[\delta_{coset}, \delta_{\varepsilon \bar{\mathcal{Q}}}^{int}] \varphi^a = -i\bar{\omega}_{mn} \varepsilon^k \sum \frac{\partial}{\partial \eta^k} \left(\eta^m \eta^n (\hat{\mathcal{U}}_1 + \hat{\mathcal{U}}_2) + \zeta^m \zeta^n (\hat{\mathcal{U}}_3 + \hat{\mathcal{U}}_4) \right) \mathcal{K}_\alpha^{a(0, \eta, \zeta)}, \quad (\text{B.14})$$

which after using (B.11) and moving the η -derivative, becomes

$$2i\bar{\omega}_{mn} \varepsilon^n \sum \eta^m \mathcal{S}^{-1} \mathcal{K}_\alpha^{a(0, \eta, \zeta)} - i\bar{\omega}_{mn} \sum (\eta^m \eta^n \mathcal{S}^{-1} + \zeta^m \zeta^n \mathcal{T}^{-1}) \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)}. \quad (\text{B.15})$$

This yields (4.38).

- Even-Odd sum Relations

The K_α 's defined in (4.44), satisfy identities which convert the even sum to the odd sum and vice versa

$$\sum_{\text{even}} \frac{\partial}{\partial \eta^m} K_\alpha = - \sum_{\text{odd}} \frac{\partial}{\partial \zeta^m} K_{\alpha+\frac{1}{2}}, \quad \sum_{\text{even}} \eta^m K_\alpha = - \sum_{\text{odd}} \zeta^m K_{\alpha-\frac{1}{2}}, \quad (\text{B.16})$$

$$\sum_{\text{odd}} \frac{\partial}{\partial \eta^m} K_\alpha = + \sum_{\text{even}} \frac{\partial}{\partial \zeta^m} K_{\alpha+\frac{1}{2}}, \quad \sum_{\text{odd}} \eta^m K_\alpha = + \sum_{\text{even}} \zeta^m K_{\alpha-\frac{1}{2}}. \quad (\text{B.17})$$

¹⁰In these appendices, it is implicitly assumed that only the terms linear in ϵ and $\bar{\varepsilon}$ are kept. In addition, whenever the sums are involved, setting $\eta = \zeta = 0$ after differentiations is also assumed.

$$\sum_{\text{even}} \left(\zeta^m \frac{\partial}{\partial \eta^n} K_\alpha - \eta^m \frac{\partial}{\partial \zeta^n} K_{\alpha+1} \right) = \delta_n^m \sum_{\text{odd}} K_{\alpha+\frac{1}{2}}, \quad (\text{B.18})$$

$$\sum_{\text{odd}} \left(\zeta^m \frac{\partial}{\partial \eta^n} K_\alpha - \eta^m \frac{\partial}{\partial \zeta^n} K_{\alpha+1} \right) = -\delta_n^m \sum_{\text{even}} K_{\alpha+\frac{1}{2}}. \quad (\text{B.19})$$

- Identities for alternate nesting of the supersymmetry parameters

$$\sum_{\text{even}} \left(E_{\bar{\varepsilon}}, E_{-\bar{\varepsilon}}(, \) \right)_\alpha = - \sum_{\text{odd}} \left(, (E_{\bar{\varepsilon}}, E_{-\bar{\varepsilon}}) \right)_{\alpha+\frac{1}{2}}, \quad (\text{B.20})$$

$$\sum_{\text{odd}} \left(E_{\bar{\varepsilon}}, E_{-\bar{\varepsilon}}(, \) \right)_\alpha = + \sum_{\text{even}} \left(, (E_{\bar{\varepsilon}}, E_{-\bar{\varepsilon}}) \right)_{\alpha+\frac{1}{2}}. \quad (\text{B.21})$$

Similar relations hold for ε .

- Identities without sums:

$$\frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \left[\left(\eta^m \frac{\partial}{\partial \eta^n} + \zeta^m \frac{\partial}{\partial \zeta^n} \right) - \frac{1}{4} \delta^m_n \left(\eta^k \frac{\partial}{\partial \eta^k} + \zeta^k \frac{\partial}{\partial \zeta^k} \right) \right] = 0 \quad (\text{B.22})$$

and

$$\begin{aligned} & \frac{d^{[4]}}{d\partial^{+2}} \left[\frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \mathcal{K}_\alpha^{a(0,\eta,\zeta)} \right]^* \\ &= \frac{\partial}{\partial \eta^{[2+2\alpha]}} \frac{\partial}{\partial \zeta^{[2-2\alpha]}} \frac{(-1)^{2\alpha}}{\partial^{+2\alpha}} \left(\partial^{+2\alpha}, \frac{1}{\partial^{+(-4\alpha)}} (\partial^{+(-2\alpha)}, \partial^{+(-2\alpha)}) \right) \mathcal{K}_\alpha^{a(0,\eta,\zeta)}, \end{aligned} \quad (\text{B.23})$$

which are valid for each $\alpha \in \{-1, -1/2, 0, +1/2, +1\}$ after setting $\eta = \zeta = 0$.

- The Recursion Relation

$$\partial^+ \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+2}} (\partial^+, \partial^+) \right)_\alpha = \left(, (, \) \right)_{\alpha+1}, \quad (\text{B.24})$$

is derived using,

$$\frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \omega^{mn} \frac{\partial}{\partial \eta^m} \frac{\partial}{\partial \eta^n} = \frac{\partial}{\partial \eta^{[4-2\alpha]}} \frac{\partial}{\partial \zeta^{[+2\alpha]}} \omega^{mn} \frac{\partial}{\partial \zeta^m} \frac{\partial}{\partial \zeta^n} \quad (\text{B.25})$$

and

$$\begin{aligned} \frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \bar{\omega}_{mn} \eta^m \eta^n &= -2\bar{\omega}_{[2]} \frac{\partial}{\partial \eta^{[-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \quad (\text{or} = 0) \\ \frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \bar{\omega}_{mn} \zeta^m \zeta^n &= -2\bar{\omega}_{[2]} \frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[+2\alpha]}}, \quad (\text{or} = 0) \end{aligned} \quad (\text{B.26})$$

where we defined $(m = 0, 2)$

$$A_{[m]} B_{[n]} C_{[4-m-n]} \equiv \frac{1}{m!n!(4-m-n)!} \varepsilon^{i_1 \dots i_4} A_{i_1 \dots i_m} B_{i_{m+1} \dots i_{m+n}} C_{i_{m+n+1} \dots i_4}. \quad (\text{B.27})$$

The identity (B.26) is valid only after setting $\eta = \zeta = 0$, and its right hand side vanishes whenever the power of the η - or ζ -derivative there comes out to be negative.

Using (B.25) together with shifting $\alpha \rightarrow \alpha + 1$ to bring the sums to common limits, we find

$$\sum_{\text{even}} \left(\frac{\partial}{\partial \eta^{mn}} \mathcal{S} + \frac{\partial}{\partial \zeta^{mn}} \mathcal{T} \right) \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)} = \sum_{\text{even}} \frac{\partial}{\partial \zeta^{mn}} \left(-\mathcal{S} \mathcal{K}_{\alpha+1}^{a(\varepsilon, \eta, \zeta)} + T \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)} \right). \quad (\text{B.28})$$

In a similar way, (B.26) implies

$$\begin{aligned} & \bar{\omega}_{mn} \sum_{\text{even}} \left(\eta^m \eta^n \mathcal{S}^{-1} + \zeta^m \zeta^n \mathcal{T}^{-1} \right) \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)}, \\ &= -2\bar{\omega}_{[2]} \sum_{\alpha=-1,0} (-1)^\alpha \frac{\partial}{\partial \eta^{[-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \left(\mathcal{S}^{-1} \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)} - \mathcal{T}^{-1} \mathcal{K}_{\alpha+1}^{a(\varepsilon, \eta, \zeta)} \right). \end{aligned} \quad (\text{B.29})$$

It is then obvious that the vanishing of (B.28) and (B.29) requires the recursion relation (4.36). The proof in the odd case is similar.

- Inside-Out-Constraint

Finally, the identity (B.23) implies that, in the even case,

$$\frac{d^{[4]}}{2\partial^{+2}} \left(\sum_{\text{even}} \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)} \right)^* = \sum_{\text{even}} \frac{1}{\partial^{+(-2\alpha)}} \left(\partial^{+(-2\alpha)}, \frac{1}{\partial^{+4\alpha}} \left(\partial^{+2\alpha}, \partial^{+2\alpha} \right) \right) \mathcal{K}_{-\alpha}^{a(\bar{\varepsilon}, \eta, \zeta)}. \quad (\text{B.30})$$

The inside-out constraint requires

$$\frac{d^{[4]}}{2\partial^{+2}} \left(\sum_{\text{even}} \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)} \right)^* = \sum_{\text{even}} \mathcal{K}_\alpha^{a(\bar{\varepsilon}, \eta, \zeta)}, \quad (\text{B.31})$$

which demands the following relations between the exponents

$$\begin{aligned} A_\alpha &= A_{-\alpha} - 2\alpha, & B_\alpha &= B_{-\alpha} - 2\alpha, & M_\alpha &= M_{-\alpha} + 4\alpha, \\ C_\alpha &= C_{-\alpha} + 2\alpha, & D_\alpha &= D_{-\alpha} + 2\alpha. \end{aligned} \quad (\text{B.32})$$

These relations, in turn, follow from the recursion relation (4.37). For example,

$$A_{\alpha+1} = A_\alpha - 1 \quad \Rightarrow \quad A_{\alpha+k} = A_\alpha - k \quad \Rightarrow \quad A_{-\alpha} = A_\alpha - (-2\alpha). \quad (\text{B.33})$$

Therefore, the recursion relation (4.37) implies that the inside-out constraint is satisfied. The same is true in the odd case.

C Calculating $\delta_{\mathcal{P}^-}^{(1)} \varphi^a$ and $\delta_{\mathcal{J}^-}^{(1)} \varphi^a$

The first commutator in (4.42) involves, in the odd case,

$$\delta_{\epsilon\overline{\mathcal{Q}}}^{free} \varphi^a = \frac{1}{\sqrt{2}} \epsilon^m \overline{q}_m \frac{\partial}{\partial^+} \varphi^a, \quad \delta_{\epsilon\overline{\mathcal{Q}}}^{int} \varphi^a = i \overline{\epsilon}_n \mathcal{S}^{-1} \sum_{\text{odd}} \eta^n \mathcal{K}_\alpha^{a(0,\eta,\zeta)}. \quad (\text{C.1})$$

Applying the master formula (B.5) with $\mathcal{O} = \overline{q}_m \partial/\partial^+$, we note that all the commutators vanish, whereas for the triplets the formula (B.7) can be applied. This gives ¹¹

$$[\delta_{\epsilon\overline{\mathcal{Q}}}^{free}, \delta_{\epsilon\overline{\mathcal{Q}}}^{int}] \varphi^a = \frac{i}{\sqrt{2}} \epsilon^m \overline{\epsilon}_n \mathcal{S}^{-1} \sum_{\text{odd}} \eta^n \left(\frac{\partial}{\partial \eta^m} \frac{\partial}{\partial r} \mathcal{S} K_\alpha^{a[r,1]} + \frac{\partial}{\partial \zeta^m} \frac{\partial}{\partial r} \mathcal{T} K_\alpha^{a[1,r]} \right), \quad (\text{C.2})$$

where (B.10) has also been used. The second commutator in (4.42) involves

$$\delta_{\epsilon\overline{\mathcal{Q}}}^{free} \varphi^a = i \overline{\epsilon}_n \theta^n \partial \varphi^a, \quad \delta_{\epsilon\overline{\mathcal{Q}}}^{int} \varphi^a = \frac{1}{\sqrt{2}} \epsilon^m \sum_{\text{odd}} \frac{\partial}{\partial \eta^m} \mathcal{K}_\alpha^{a(0,\eta,\zeta)}. \quad (\text{C.3})$$

With $\mathcal{O} = \theta^n \partial$, both triplets in the master formula (B.5) vanish. Using

$$[E_\eta, \theta^n \partial] = \eta^n \widehat{\partial} E_\eta \quad (\text{C.4})$$

and the identity (B.9), we find

$$[\delta_{\epsilon\overline{\mathcal{Q}}}^{free}, \delta_{\epsilon\overline{\mathcal{Q}}}^{int}] \varphi^a = \frac{i}{\sqrt{2}} \overline{\epsilon}_n \epsilon^m \sum_{\text{odd}} \frac{\partial}{\partial \eta^m} \frac{\partial}{\partial r} \left(\eta^n K_\alpha^{a[r,1]} + \zeta^n K_\alpha^{a[1,r]} \right). \quad (\text{C.5})$$

Therefore,

$$\begin{aligned} [\delta_{\epsilon\overline{\mathcal{Q}}}^{free}, \delta_{\epsilon\overline{\mathcal{Q}}}^{int}] \varphi^a - [\delta_{\epsilon\overline{\mathcal{Q}}}^{free}, \delta_{\epsilon\overline{\mathcal{Q}}}^{int}] \varphi^a &= \frac{i}{\sqrt{2}} \epsilon^m \overline{\epsilon}_n \frac{\partial}{\partial r} \sum_{\text{odd}} \left(\eta^n \frac{\partial}{\partial \eta^m} + \frac{\partial}{\partial \eta^m} \eta^n \right) K_\alpha^{a[r,1]} \\ &\quad + \frac{i}{\sqrt{2}} \epsilon^m \overline{\epsilon}_n \frac{\partial}{\partial r} \sum_{\text{odd}} \left(\eta^n \frac{\partial}{\partial \zeta^m} K_{\alpha+1}^{a[1,r]} - \zeta^n \frac{\partial}{\partial \eta^m} K_\alpha^{a[1,r]} \right). \end{aligned} \quad (\text{C.6})$$

Using the identity (B.19), this becomes

$$[\delta_{\epsilon\overline{\mathcal{Q}}}^{free}, \delta_{\epsilon\overline{\mathcal{Q}}}^{int}] \varphi^a - [\delta_{\epsilon\overline{\mathcal{Q}}}^{free}, \delta_{\epsilon\overline{\mathcal{Q}}}^{int}] \varphi^a = \frac{i}{\sqrt{2}} \epsilon^m \overline{\epsilon}_m \frac{\partial}{\partial r} \left(\sum_{\text{odd}} K_\alpha^{a[r,1]} + \sum_{\text{even}} K_{\alpha+\frac{1}{2}}^{a[1,r]} \right) \quad (\text{C.7})$$

so that the $O(f)$ part of the Hamiltonian shift in the odd case is

$$\delta_{\mathcal{P}^-}^{(1)\text{odd}} \varphi^a = -\frac{i}{2} \frac{\partial}{\partial r} \left(\sum_{\text{odd}} K_\alpha^{a[r,1]} + \sum_{\text{even}} K_{\alpha+\frac{1}{2}}^{a[1,r]} \right). \quad (\text{C.8})$$

¹¹Setting $r = 0$ after the differentiation is kept implicit.

which reproduces (4.43). In the even case, because of (B.18), the corresponding expression has the relative *minus* sign, which explains the rule (4.46).

The $O(f)$ part of the Lorentz boost follows from commuting with the kinematical special conformal transformation K ,

$$[\delta_K, \boldsymbol{\delta}_{\mathcal{P}^-}^{(1)}] \varphi^a = 2i \boldsymbol{\delta}_{\mathcal{J}^-}^{(1)} \varphi^a, \quad (\text{C.9})$$

with $\delta_K \varphi^a = 2ix\mathcal{A}\varphi^a$. Defining $\delta_{x\mathcal{A}} \varphi^a = x\mathcal{A}\varphi^a$, we have

$$\boldsymbol{\delta}_{\mathcal{J}^-}^{(1)\text{odd}} \varphi^a = -\frac{i}{2} \frac{\partial}{\partial r} \left(\sum_{\text{odd}} [\delta_{x\mathcal{A}}, K_\alpha^{a[r,1]}] + \sum_{\text{even}} [\delta_{x\mathcal{A}}, K_{\alpha+\frac{1}{2}}^{a[1,r]}] \right). \quad (\text{C.10})$$

In calculating the commutator part of the master formula (B.5), we use

$$[E_r E_\eta \partial^{+k}, x\mathcal{A}] = \left(x \left[\frac{1}{2} \eta \frac{\partial}{\partial \eta} - k \right] + rx^- + r \left[\frac{1}{2} \left(\eta \frac{\partial}{\partial \eta} - \mathcal{N} \right) + \frac{3}{2} - k \right] \frac{1}{\partial^+} \right) E_r E_\eta \partial^{+k}. \quad (\text{C.11})$$

The x^- -dependent contributions cancel because E_r comes with E_{-r} . For the x -dependent contributions, we note that moving x to the left involves

$$[E_r, x] = r \frac{1}{\partial^+} E_r. \quad (\text{C.12})$$

As $x\mathcal{A}$ is a derivative operator *plus* $x/2$, the contribution from the triplets in (B.5) is

$$\begin{aligned} \left(\left\{ x\mathcal{A} \right\}_{12} + \left\{ x\mathcal{A} \right\}_{34} \right) K_\alpha^{a[r,1]} &= \left(x - \frac{r}{2} \widehat{\mathcal{U}}_2 \right) K_\alpha^{a[r,1]}, \\ \left(\left\{ x\mathcal{A} \right\}_{12} + \left\{ x\mathcal{A} \right\}_{34} \right) K_\alpha^{a[1,r]} &= x K_\alpha^{a[1,r]}. \end{aligned} \quad (\text{C.13})$$

Alltogether, we find

$$\begin{aligned} [\delta_{x\mathcal{A}}, K_\alpha^{a[r,1]}] &= \left\{ -x + \frac{r}{2} (\mathcal{O}_\eta \widehat{\mathcal{U}}_1 - \mathcal{O}_\eta \widehat{\mathcal{U}}_2) + r \left(\frac{3}{2} - B_\alpha \right) \widehat{\mathcal{U}}_1, \right. \\ &\quad \left. -r (M_\alpha - C_\alpha - D_\alpha + 3 + \alpha) \widehat{\mathcal{U}}_2 \right\} K_\alpha^{a[r,1]} \\ [\delta_{x\mathcal{A}}, K_\alpha^{a[1,r]}], &= \left\{ -x + \frac{r}{2} (\mathcal{O}_\zeta \widehat{\mathcal{U}}_3 - \mathcal{O}_\zeta \widehat{\mathcal{U}}_4) + r \left(\frac{3}{2} - C_\alpha \right) \widehat{\mathcal{U}}_3 - r \left(\frac{3}{2} - D_\alpha \right) \widehat{\mathcal{U}}_4 \right\} K_\alpha^{a[1,r]}, \end{aligned} \quad (\text{C.14})$$

where we defined

$$\mathcal{O}_\eta \equiv \eta \frac{\partial}{\partial \eta} - \mathcal{N}, \quad \mathcal{O}_\zeta \equiv \zeta \frac{\partial}{\partial \zeta} - \mathcal{N} \quad (\text{C.15})$$

and used that, when η - and ζ -derivatives act on the *whole* $K_\alpha^a \equiv \mathcal{K}_\alpha^{a(0,\eta,\zeta)}$, we have

$$\eta \frac{\partial}{\partial \eta} = 2 - 2\alpha, \quad \zeta \frac{\partial}{\partial \zeta} = 2 + 2\alpha \quad (\text{C.16})$$

according to the definition of the sums in (4.32) and (4.35). It then follows that ¹²

$$\begin{aligned}
\delta_{\mathcal{J}^-}^{(1)\text{odd}} \varphi^a &= -x \delta_{\mathcal{P}^-}^{(1)\text{odd}} \varphi^a - \frac{i}{2} \left\{ \frac{1}{2} \frac{\partial}{\partial u} \left(\sum_{\text{odd}} K_\alpha^a \{u, 1\} + \sum_{\text{even}} K_{\alpha+\frac{1}{2}}^a \{1, u\} \right) \right. \\
&\quad - \sum_{\text{odd}} \left[\left(B_\alpha - \frac{3}{2} \right) \widehat{\mathcal{U}}_1 + (M_\alpha - C_\alpha - D_\alpha + 3 + \alpha) \widehat{\mathcal{U}}_2 \right] K_\alpha^a \\
&\quad \left. - \sum_{\text{even}} \left[\left(C_{\alpha+\frac{1}{2}} - \frac{3}{2} \right) \widehat{\mathcal{U}}_3 - \left(D_{\alpha+\frac{1}{2}} - \frac{3}{2} \right) \widehat{\mathcal{U}}_4 \right] K_{\alpha+\frac{1}{2}}^a, \right\} \tag{C.17}
\end{aligned}$$

where we defined

$$K_\alpha^a \{u, 1\} \equiv (E_{u,\eta} \mathcal{U}_1)(E_{-u,\eta} \mathcal{U}_2) K_\alpha^a, \quad K_\alpha^a \{1, u\} \equiv (E_{u,\zeta} \mathcal{U}_3)(E_{-u,\zeta} \mathcal{U}_4) K_\alpha^a \tag{C.18}$$

with

$$E_{u,\eta} \equiv e^{u\widehat{\mathcal{O}}_\eta}, \quad E_{u,\zeta} \equiv e^{u\widehat{\mathcal{O}}_\zeta}. \tag{C.19}$$

The result in the even case is obtained by the substitution (4.46).

D Calculating $[\delta_{\mathcal{P}^-}, \delta_{\mathcal{J}^-}] \varphi^a$

The commutator of the Hamiltonian shift $\delta_{\mathcal{P}^-} \varphi^a$ with the Lorentz boost $\delta_{\mathcal{J}^-} \varphi^a$ is

$$[\delta_{\mathcal{P}^-}, \delta_{\mathcal{J}^-}] \varphi^a = [\delta_{\mathcal{P}^-}^{free}, \delta_{\mathcal{J}^-}^{(1)}] \varphi^a + [\delta_{\mathcal{P}^-}^{(1)}, \delta_{\mathcal{J}^-}^{free}] \varphi^a + O(f^2), \tag{D.1}$$

where

$$\delta_{\mathcal{P}^-}^{free} \varphi^a = -\frac{i}{2} \frac{\partial^2}{\partial^+} \varphi^a, \quad \delta_{\mathcal{J}^-}^{free} \varphi^a = -i \frac{\partial}{\partial^+} \mathcal{A} \varphi^a \tag{D.2}$$

and $\delta_{\mathcal{P}^-}^{(1)} \varphi^a$ with $\delta_{\mathcal{J}^-}^{(1)} \varphi^a$, in the odd case, are given in (C.8) and (C.17).

In $[\delta_{\mathcal{P}^-}^{free}, \delta_{\mathcal{J}^-}^{(1)}] \varphi^a$, only the the first term in (C.17), with explicit x , contributes to the commutator part of the master formula (B.5). For the triplets in (B.5), we can use (B.7) which gives

$$[\delta_{\mathcal{P}^-}^{free}, K_\alpha^a] = -\frac{i}{2} \mathcal{S} \frac{\partial^2}{\partial r \partial r'} \left(K_\alpha^{a[r+r',1]} + K_{\alpha+1}^{a[1,r+r']} \right), \tag{D.3}$$

¹²Setting $u = 0$ after the differentiation is kept implicit.

where also the recursion relation (4.36) has been used. It then immediately follows that

$$\begin{aligned}
[\delta_{\mathcal{P}^-}^{free}, \delta_{\mathcal{J}^-}^{(1) odd}] \varphi^a &= -x[\delta_{\mathcal{P}^-}^{free}, \delta_{\mathcal{P}^-}^{(1) odd}] \varphi^a - i \frac{\partial}{\partial^+} \delta_{\mathcal{P}^-}^{(1) odd} \varphi^a - \frac{1}{4} \mathcal{S} \frac{\partial^2}{\partial r \partial r'} \left\{ \right. \\
&\quad - \sum_{\text{odd}} \left[\left(B_\alpha - \frac{3}{2} \right) \widehat{\mathcal{U}}_1 + (M_\alpha - C_\alpha - D_\alpha + 3 + \alpha) \widehat{\mathcal{U}}_2 \right] \left(K_\alpha^{a[r+r',1]} + K_{\alpha+1}^{a[1,r+r']} \right) \\
&\quad - \sum_{\text{even}} \left[\left(C_{\alpha+\frac{1}{2}} - \frac{3}{2} \right) \widehat{\mathcal{U}}_3 - \left(D_{\alpha+\frac{1}{2}} - \frac{3}{2} \right) \widehat{\mathcal{U}}_4 \right] \left(K_{\alpha+\frac{1}{2}}^{a[r+r',1]} + K_{\alpha+\frac{3}{2}}^{a[1,r+r']} \right) \\
&\quad \left. + \frac{1}{2} \frac{\partial}{\partial u} \left[\sum_{\text{odd}} \left(K_\alpha^{a[r+r',1]\{u,1\}} + K_{\alpha+1}^{a[1,r+r']\{u,1\}} \right) + \sum_{\text{even}} \left(K_{\alpha+\frac{1}{2}}^{a[r+r',1]\{1,u\}} + K_{\alpha+\frac{3}{2}}^{a[1,r+r']\{1,u\}} \right) \right] \right\}. \quad (\text{D.4})
\end{aligned}$$

The commutator $[\delta_{\mathcal{J}^-}^{free}, \delta_{\mathcal{P}^-}^{(1)}] \varphi^a$ requires longer analysis. First, we apply the master formula (B.5) noting that

$$[E_r E_\eta \partial^{+k}, \frac{\partial}{\partial^+} \mathcal{A}] = \frac{\partial}{\partial^+} \left(\frac{1}{2} r \frac{\partial}{\partial r} + \frac{1}{2} \eta \frac{\partial}{\partial \eta} - k \right) E_r E_\eta \partial^{+k}. \quad (\text{D.5})$$

For the triplets in (B.5), we have

$$\left\{ \frac{\partial}{\partial^+} \mathcal{A} \right\} = \left\{ (\mathcal{A} + \frac{1}{2}) \frac{\partial}{\partial^+} \right\} = -\frac{1}{2} \left\{ x \frac{\partial^2}{\partial^+} \right\} - \frac{1}{2} \left\{ \mathcal{N} \widehat{\partial} \right\} + \left\{ \widehat{\partial} \right\}, \quad (\text{D.6})$$

where the x^- -dependent part dropped out. For the x -dependent part, we observe that

$$\begin{aligned}
\left(\left\{ x \frac{\partial^2}{\partial^+} \right\}_{12} + \left\{ x \frac{\partial^2}{\partial^+} \right\}_{34} \right) K_\alpha^{a[r,1]} &= 2ix \left[\delta_{\mathcal{P}^-}^{free}, K_\alpha^{a[r,1]} \right] - r \widehat{\mathcal{U}}_2 \left\{ \frac{\partial^2}{\partial^+} \right\}_{34} K_\alpha^{a[r,1]} \\
\left(\left\{ x \frac{\partial^2}{\partial^+} \right\}_{12} + \left\{ x \frac{\partial^2}{\partial^+} \right\}_{34} \right) K_\alpha^{a[1,r]} &= 2ix \left[\delta_{\mathcal{P}^-}^{free}, K_\alpha^{a[1,r]} \right], \quad (\text{D.7})
\end{aligned}$$

where, using (B.7) and the recursion relation (4.36), we also have

$$\left\{ \frac{\partial^2}{\partial^+} \right\}_{34} K_\alpha^{a[r,1]} = \frac{\partial}{\partial r'} \frac{\partial}{\partial r''} \mathcal{S} K_{\alpha+1}^{a[r,r'+r'']}. \quad (\text{D.8})$$

For the $\eta \frac{\partial}{\partial \eta}$ and $\zeta \frac{\partial}{\partial \zeta}$ parts arising from (D.5), we use

$$\begin{aligned}
\eta \frac{\partial}{\partial \eta} \widehat{\mathcal{U}}_1 + \eta \frac{\partial}{\partial \eta} \widehat{\mathcal{U}}_2 &= \left\{ \eta \frac{\partial}{\partial \eta} \widehat{\partial} \right\}_{12} + (2 - 2\alpha) \widehat{\partial}, \\
\zeta \frac{\partial}{\partial \zeta} \widehat{\mathcal{U}}_3 + \zeta \frac{\partial}{\partial \zeta} \widehat{\mathcal{U}}_4 &= \left\{ \zeta \frac{\partial}{\partial \zeta} \widehat{\partial} \right\}_{34} + (2 + 2\alpha) \widehat{\partial} \mathcal{U}_2. \quad (\text{D.9})
\end{aligned}$$

These triplets then combine with the \mathcal{N} -dependent triplets in (D.6), so that the combinations (C.15) form, and we can use

$$\left\{ \mathcal{O}_\eta \widehat{\partial} \right\}_{12} K_\alpha^a = \mathcal{S} \frac{\partial}{\partial u} \frac{\partial}{\partial r'} K_\alpha^{a[r',1]\{u,1\}}, \quad \left\{ \mathcal{O}_\zeta \widehat{\partial} \right\}_{34} K_\alpha^a = \mathcal{S} \frac{\partial}{\partial u} \frac{\partial}{\partial r'} K_{\alpha+1}^{a[1,r']\{1,u\}}. \quad (\text{D.10})$$

Finally, using the following ‘‘recombination’’ identity,

$$\begin{aligned} & \left(\lambda_1 \widehat{\mathcal{U}}_1 + \lambda_2 \widehat{\mathcal{U}}_2 + \lambda_3 \widehat{\mathcal{U}}_3 + \lambda_4 \widehat{\mathcal{U}}_4 \right) K_\alpha^a = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \widehat{\mathcal{K}}_a \\ & + \mathcal{S} \frac{\partial}{\partial r'} \left\{ \left[\lambda_1 \widehat{\mathcal{U}}_1 - (\lambda_2 + \lambda_3 + \lambda_4) \widehat{\mathcal{U}}_2 \right] K_\alpha^{a[r',1]} + \left(\lambda_3 \widehat{\mathcal{U}}_3 - \lambda_4 \widehat{\mathcal{U}}_4 \right) K_{\alpha+1}^{a[1,r']} \right\} \end{aligned} \quad (\text{D.11})$$

we obtain

$$\begin{aligned} [\delta_{\mathcal{P}^-}^{free}, \delta_{\mathcal{P}^-}^{(1) odd}] \varphi^a &= -x[\delta_{\mathcal{P}^-}^{free}, \delta_{\mathcal{P}^-}^{(1) odd}] \varphi^a - i \frac{\partial}{\partial^+} \delta_{\mathcal{P}^-}^{(1) odd} \varphi^a - \frac{1}{4} \mathcal{S} \frac{\partial^2}{\partial r \partial r'} \left\{ \sum_{\text{odd}} \widehat{\mathcal{U}}_2 K_{\alpha+1}^{a[1,r+r']} \right. \\ &+ \sum_{\text{odd}} \left[(3 - 2B_\alpha) \widehat{\mathcal{U}}_1 - (2(M_\alpha - C_\alpha - D_\alpha) + 7 + 2\alpha) \widehat{\mathcal{U}}_2 \right] K_\alpha^{a[r+r',1]} \\ &+ \sum_{\text{odd}} \left[(2 - 2C_\alpha) \widehat{\mathcal{U}}_3 - (2 - 2D_\alpha) \widehat{\mathcal{U}}_4 \right] K_{\alpha+1}^{a[r,r']} \\ &+ \sum_{\text{even}} \left[(2 - 2B_{\alpha+\frac{1}{2}}) \widehat{\mathcal{U}}_1 - (2(M_{\alpha+\frac{1}{2}} - C_{\alpha+\frac{1}{2}} - D_{\alpha+\frac{1}{2}}) + 8 + 2\alpha) \widehat{\mathcal{U}}_2 \right] K_{\alpha+\frac{1}{2}}^{a[r,r']} \\ &+ \sum_{\text{even}} \left[(3 - 2C_{\alpha+\frac{1}{2}}) \widehat{\mathcal{U}}_3 - (3 - 2D_{\alpha+\frac{1}{2}}) \widehat{\mathcal{U}}_4 \right] K_{\alpha+\frac{3}{2}}^{a[1,r+r']} \\ &+ \left. \frac{\partial}{\partial u} \left[\sum_{\text{odd}} \left(K_\alpha^{a[r+r',1]\{u,1\}} + K_{\alpha+1}^{a[r,r']\{1,u\}} \right) + \sum_{\text{even}} \left(K_{\alpha+\frac{1}{2}}^{a[r,r']\{u,1\}} + K_{\alpha+\frac{3}{2}}^{a[1,r+r']\{1,u\}} \right) \right] \right\}. \end{aligned} \quad (\text{D.12})$$

The $O(f)$ part of (D.1) is the difference of (D.4) and (D.12). We see that the first two terms on the right hand side of (D.12) cancel the corresponding terms in (D.4). Using the following identities,

$$\begin{aligned} \frac{\partial}{\partial u} \left(\sum_{\text{even}} K_\alpha^{a\{u,1\}} + \sum_{\text{odd}} K_{\alpha+\frac{1}{2}}^{a\{1,u\}} \right) &= 4 \sum_{\text{even}} \widehat{\mathcal{U}}_2 K_\alpha^a + \sum_{\text{even}} (2 - 2\alpha) (\widehat{\mathcal{U}}_1 - \widehat{\mathcal{U}}_2) K_\alpha^a \\ &+ \sum_{\text{odd}} (2 + 2\alpha) (\widehat{\mathcal{U}}_3 - \widehat{\mathcal{U}}_4) K_{\alpha+\frac{1}{2}}^a \\ \frac{\partial}{\partial u} \left(\sum_{\text{odd}} K_\alpha^{a\{u,1\}} - \sum_{\text{even}} K_{\alpha+\frac{1}{2}}^{a\{1,u\}} \right) &= 4 \sum_{\text{odd}} \widehat{\mathcal{U}}_2 K_\alpha^a + \sum_{\text{odd}} (2 - 2\alpha) (\widehat{\mathcal{U}}_1 - \widehat{\mathcal{U}}_2) K_\alpha^a \\ &- \sum_{\text{even}} (2 + 2\alpha) (\widehat{\mathcal{U}}_3 - \widehat{\mathcal{U}}_4) K_{\alpha+\frac{1}{2}}^a \end{aligned} \quad (\text{D.13})$$

we find that the contribution to the difference of (D.4) and (D.12) from the terms with $\frac{\partial}{\partial u}$ is

$$\begin{aligned} -\frac{1}{4} \mathcal{S} \frac{\partial^2}{\partial r \partial r'} \left\{ & - \sum_{\text{odd}} \left[(1 - \alpha) \widehat{\mathcal{U}}_1 + (1 + \alpha) \widehat{\mathcal{U}}_2 \right] K_\alpha^{a[r+r',1]} + \sum_{\text{even}} (1 + \alpha) (\widehat{\mathcal{U}}_3 - \widehat{\mathcal{U}}_4) K_{\alpha+\frac{1}{2}}^{a[r+r',1]} \\ & + \sum_{\text{odd}} \left[(1 - \alpha) \widehat{\mathcal{U}}_1 + (1 + \alpha) \widehat{\mathcal{U}}_2 \right] K_{\alpha+1}^{a[1,r+r']} - \sum_{\text{even}} (1 + \alpha) (\widehat{\mathcal{U}}_3 - \widehat{\mathcal{U}}_4) K_{\alpha+\frac{3}{2}}^{a[1,r+r']} \\ & - 2 \sum_{\text{even}} \left[(1 - \alpha) \widehat{\mathcal{U}}_1 + (1 + \alpha) \widehat{\mathcal{U}}_2 \right] K_{\alpha+\frac{1}{2}}^{a[r,r']} - 2 \sum_{\text{odd}} (1 + \alpha) (\widehat{\mathcal{U}}_3 - \widehat{\mathcal{U}}_4) K_{\alpha+1}^{a[r,r']} \right\}. \end{aligned} \quad (\text{D.14})$$

The complete expression for the difference of (D.4) and (D.12) then easily follows, and we find that the exponents B_α , M_α , C_α , D_α appear in combinations with explicit α 's that are α -*independent* thanks to the recursion relations (4.37). The final answer for the $O(f)$ part of the commutator (D.1) is given in equations (4.59) through (4.62).

E Odd Ansatz: the BLG Solution

As we mentioned in the text, the commutator $[\delta_{\mathcal{P}-}, \delta_{\mathcal{J}-}] \varphi^a$ contains two powers of the transverse derivative and four powers of \bar{d} . A necessary condition for its vanishing is that the terms for which $\partial^2 \bar{d}^{[4]}$ act on the same superfield vanish by themselves. In this appendix, we single those terms out for the odd Ansatz.

There are two terms where $\partial^2 \bar{d}^{[4]}$ acts on the “first” superfield,

$$(C_{-\frac{1}{2}} - \frac{3}{2}) \frac{1}{\partial^+} \left(\frac{\partial^2 \bar{d}^{[4]}}{\partial^{+5}}, \partial^+ \left(\frac{1}{\partial^+}, \right) \right) - (D_{-\frac{1}{2}} - \frac{3}{2}) \frac{1}{\partial^+} \left(\frac{\partial^2 \bar{d}^{[4]}}{\partial^{+5}}, \partial^+ \left(, \frac{1}{\partial^+} \right) \right), \quad (\text{E.1})$$

which must be cancelled by terms where $\partial^2 \bar{d}^{[4]}$ acts on the “second” and “third” superfields. If we require that the terms with the most inverse powers of delplus (most singular) vanish by themselves, we arrive at

$$M_{-\frac{1}{2}} = 2(C_{-\frac{1}{2}} + D_{-\frac{1}{2}}) - 6 \leq -2, \quad B_{-\frac{1}{2}} = C_{-\frac{1}{2}} - M_{-\frac{1}{2}}. \quad (\text{E.2})$$

These terms reduce to (dropping the subscript $-\frac{1}{2}$) and we will write $\bar{d}^{[4]}$ as \bar{d}^4 in the remaining calculations in this and the remaining appendices to get the expressions more transparent.

$$\begin{aligned} & 2(B-3) \left[-2 \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+2}} \left(, \frac{\partial^2 \bar{d}^4}{\partial^{+2}} \right) \right) + \partial^+ \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+2}} \left(, \frac{\partial^2 \bar{d}^4}{\partial^{+3}} \right) \right) + \frac{1}{\partial^+} \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+2}} \left(, \frac{\partial^2 \bar{d}^4}{\partial^+} \right) \right) \right] \\ & - (C+D-3) \left[-\partial^{+2} \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+2}} \left(, \frac{\partial^2 \bar{d}^4}{\partial^{+4}} \right) \right) + 3\partial^+ \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+2}} \left(, \frac{\partial^2 \bar{d}^4}{\partial^{+3}} \right) \right) \right. \\ & \quad \left. - 3 \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+2}} \left(, \frac{\partial^2 \bar{d}^4}{\partial^{+2}} \right) \right) + \frac{1}{\partial^+} \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+2}} \left(, \frac{\partial^2 \bar{d}^4}{\partial^+} \right) \right) \right] \\ & + (C-\frac{3}{2}) \left[-\partial^{+2} \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^+} \left(\frac{1}{\partial^+}, \frac{\partial^2 \bar{d}^4}{\partial^{+4}} \right) \right) + 3\partial^+ \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^+} \left(\frac{1}{\partial^+}, \frac{\partial^2 \bar{d}^4}{\partial^{+3}} \right) \right) \right. \\ & \quad \left. - 3 \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^+} \left(\frac{1}{\partial^+}, \frac{\partial^2 \bar{d}^4}{\partial^{+2}} \right) \right) + \frac{1}{\partial^+} \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^+} \left(\frac{1}{\partial^+}, \frac{\partial^2 \bar{d}^4}{\partial^+} \right) \right) \right. \\ & \quad \left. + 2\partial^+ \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+2}} \left(\frac{1}{\partial^+}, \frac{\partial^2 \bar{d}^4}{\partial^{+3}} \right) \right) - 4 \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+2}} \left(\frac{1}{\partial^+}, \frac{\partial^2 \bar{d}^4}{\partial^{+2}} \right) \right) + 2 \frac{1}{\partial^+} \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+2}} \left(\frac{1}{\partial^+}, \frac{\partial^2 \bar{d}^4}{\partial^+} \right) \right) \right], \end{aligned} \quad (\text{E.3})$$

where we used

$$(C - 3/2)\left(\frac{1}{\partial^+}, \frac{\partial^2 \bar{d}^4}{\partial^{+a}}\right) - (D - 3/2)\left(, \frac{\partial^2 \bar{d}^4}{\partial^{+(a+1)}}\right) = \\ (C - 3/2)\partial^+\left(\frac{1}{\partial^+}, \frac{\partial^2 \bar{d}^4}{\partial^{+(a+1)}}\right) - (C + D - 3)\left(, \frac{\partial^2 \bar{d}^4}{\partial^{+(a+1)}}\right). \quad (\text{E.4})$$

One then sees that many terms can vanish due to antisymmetry. For example, the first three lines in (E.3) vanish as long as $f^a_{bcd} = -f^a_{cbd}$ and $M = -2$.

In order to investigate the possible cases for $M \leq -2$, we set $M = -2 - m$ ($m \geq 0$) and $B = C - M = C + 2 - m$. When $m = 0$, (E.3) is rewritten as

$$- (C - \frac{3}{2}) \frac{f^a_{bcd}}{\partial^{+(A+1)}} \cdot \\ \left[\partial^{+3} (\partial^{+C} \varphi^b \partial^{+(C-1)} \varphi^c \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d) - 3 \partial^{+2} (\partial^{+C} \varphi^b \partial^{+(C-1)} \varphi^c \partial^2 \bar{d}^4 \partial^{+(D-2)} \varphi^d) \right. \\ + 3 \partial^+ (\partial^{+C} \varphi^b \partial^{+(C-1)} \varphi^c \partial^2 \bar{d}^4 \partial^{+(D-1)} \varphi^d) - (\partial^{+C} \varphi^b \partial^{+(C-1)} \varphi^c \partial^2 \bar{d}^4 \partial^{+D} \varphi^d) \\ - 2 \partial^{+2} (\partial^{+(C+1)} \varphi^b \partial^{+(C-1)} \varphi^c \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d) + 4 \partial^+ (\partial^{+(C+1)} \varphi^b \partial^{+(C-1)} \varphi^c \partial^2 \bar{d}^4 \partial^{+(D-2)} \varphi^d) \\ \left. - 2 (\partial^{+(C+1)} \varphi^b \partial^{+(C-1)} \varphi^c \partial^2 \bar{d}^4 \partial^{+(D-1)} \varphi^d) \right], \quad (\text{E.5})$$

which can be reorganized along terms of the form $\partial^2 \bar{d}^4 \partial^{+(D-n)} \varphi^d$ ($n = 1, 2, 3$), yielding that all cancel, except for

$$- (C - \frac{3}{2}) f^a_{bcd} \left[(\partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d) \partial^{+2} (\partial^{+(C+1)} \varphi^b \partial^{+(C-1)} \varphi^c) \right]. \quad (\text{E.6})$$

We then compare this to (E.1) when $B = C + 2$, which is

$$+ (C - \frac{3}{2}) f^a_{bcd} \left[(\partial^2 \bar{d}^4 \partial^{+(C-3)} \varphi^b) \partial^{+3} (\partial^{+(C-1)} \varphi^c \partial^{+(D)} \varphi^d) \right]. \quad (\text{E.7})$$

These two terms cancel if $f^a_{bcd} = -f^a_{dcb}$ and $C = D$. In a similar way, it is not difficult to see that the $D - 3/2$ term of (E.1) is also cancelled by the contributions that $\partial^2 \bar{d}^4$ acts on the “second” superfield $\partial^2 \bar{d}^4 \partial^{+(C-3)} \varphi^c$. It follows from (E.2) that

$$A = 3, \quad B = 3, \quad M = -2, \quad C = D = 1, \quad \text{and} \quad f^a_{[bcd]}, \quad (\text{E.8})$$

which are the exponents for the BLG solution (4.67).

When $m \neq 0$, we find for the most “singular” term

$$\partial^{+2} \left\{ \left[(\partial^{+(C-m)} \varphi^b \partial^{+(C+m)} \varphi^c - \partial^{+(C-m+1)} \varphi^b \partial^{+(C+m-1)} \varphi^c \right] \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \right\} . \quad (\text{E.9})$$

which must cancel against (E.1). However, these terms cannot cancel and no solution exists when $m \neq 0$.

F Even Ansatz: no BLG Solution

For the even case, the commutator of the Hamiltonian with the boost is given by

$$[\delta_{\mathcal{P}^-}^{\text{even}}, \delta_{\mathcal{J}^-}^{\text{even}}] \varphi^a = -\frac{1}{4} \mathcal{S} \frac{\partial^2}{\partial r \partial r'} (\mathcal{F} \mathcal{O}_1^{a, \text{even}} + \mathcal{G} \mathcal{O}_2^{a, \text{even}})_{r=r'=0} + O(f^2) , \quad (\text{F.1})$$

where

$$\begin{aligned} \mathcal{F} &\equiv (B_{-1} - \frac{7}{2}) \hat{\mathcal{U}}_1 + (M_{-1} - C_{-1} - D_{-1} + 3) \hat{\mathcal{U}}_2 , \\ \mathcal{G} &\equiv (C_{-1} - 1) \hat{\mathcal{U}}_3 - (D_{-1} - 1) \hat{\mathcal{U}}_4 , \end{aligned} \quad (\text{F.2})$$

and

$$\begin{aligned} \mathcal{O}_1^{\text{even}} &= \sum_{\text{even}} (K_{\alpha}^{[r+r',1]} - K_{\alpha+1}^{[r+r',1]}) - 2 \sum_{\text{odd}} K_{\alpha+\frac{1}{2}}^{[r,r']} , \\ \mathcal{O}_2^{\text{even}} &= \sum_{\text{odd}} (K_{\alpha+\frac{1}{2}}^{[1,r+r']} - K_{\alpha+\frac{3}{2}}^{[1,r+r']}) + 2 \sum_{\text{even}} K_{\alpha+1}^{[r,r']} . \end{aligned} \quad (\text{F.3})$$

We now search for solutions with integer-valued exponents. We show below that no such solutions exist for the even case, unlike the odd case.

We first express the r.h.s. of (F.1) in the base where $\alpha = -1$, and drop the subscripts. As done in the odd case, we only consider the terms of $\partial^2 \bar{d}^4$ acting on the same superfield. When $\partial^2 \bar{d}^4$ acts on the “first” superfield, we have

$$-\mathcal{F} \frac{1}{\partial^+} \left(\frac{\partial^2 \bar{d}^4}{\partial^{+5}}, \partial^+ (,) \right) = -(B - \frac{7}{2}) \frac{1}{\partial^+} \left(\frac{\partial^2 \bar{d}^4}{\partial^{+6}}, \partial^+ (,) \right) - (M - C - D + 3) \frac{1}{\partial^+} \left(\frac{\partial^2 \bar{d}^4}{\partial^{+5}}, (,) \right) . \quad (\text{F.4})$$

Notice that only the first term has the singular structure of $\frac{1}{\partial^{+6}}$, which is different from the odd case whose singular structure lies on both terms.

The terms with $\partial^2 \bar{d}^4$ on the “third” superfield are given by the sum of the \mathcal{F} -terms and \mathcal{G} -terms

$$\begin{aligned}
& + \mathcal{F} \frac{1}{\partial^+} \left[- \left(\partial^+, \frac{1}{\partial^{+5}} (, \partial^2 \bar{d}^4) \right) - \partial^+ \left(, \frac{1}{\partial^{+5}} (\partial^+, \frac{\partial^2 \bar{d}^4}{\partial^+}) \right) \right. \\
& \quad \left. + \partial^{+2} \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+5}} (\partial^{+2}, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) + \partial^{+3} \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+5}} (\partial^{+3}, \frac{\partial^2 \bar{d}^4}{\partial^{+3}}) \right) \right] \\
& + 2\mathcal{G} \frac{1}{\partial^+} \left[+ \partial^+ \left(, \frac{1}{\partial^{+6}} (\partial^+, \partial^2 \bar{d}^4) \right) + 2\partial^{+2} \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+6}} (\partial^{+2}, \frac{\partial^2 \bar{d}^4}{\partial^+}) \right) + \partial^{+3} \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+6}} (\partial^{+3}, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) \right] ,
\end{aligned} \tag{F.5}$$

which can also be written as

$$\begin{aligned}
& + \mathcal{F} \frac{1}{\partial^+} \left[- 2 \left(\partial^+, \frac{1}{\partial^{+3}} (, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) - 4 \left(, \frac{1}{\partial^{+3}} (\partial^+, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) - 2 \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+3}} (\partial^{+2}, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) \right. \\
& \quad + \left(\partial^+, \frac{1}{\partial^{+2}} (, \frac{\partial^2 \bar{d}^4}{\partial^{+3}}) \right) + 3 \left(, \frac{1}{\partial^{+2}} (\partial^+, \frac{\partial^2 \bar{d}^4}{\partial^{+3}}) \right) \\
& \quad \left. + 3 \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+2}} (\partial^{+2}, \frac{\partial^2 \bar{d}^4}{\partial^{+3}}) \right) + \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+2}} (\partial^{+3}, \frac{\partial^2 \bar{d}^4}{\partial^{+3}}) \right) \right] \\
& + 2\mathcal{G} \left[\left(, \frac{1}{\partial^{+4}} (\partial^+, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) + 2 \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+4}} (\partial^{+2}, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) + \left(\frac{1}{\partial^{+2}}, \frac{1}{\partial^{+4}} (\partial^{+3}, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) \right] . \tag{F.6}
\end{aligned}$$

In order to make sure that the most singular terms lie on φ^d (not on φ^c), we assume that $C > D$. Then we follow singular terms with $(\dots, \frac{1}{\partial^{+n}}(\dots, \dots))$ structure. The most singular part reads

$$-2(M - 2(C + D) + 5) \frac{1}{\partial^+} \left[\left(\partial^+, \frac{1}{\partial^{+4}} (, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) + 2 \left(, \frac{1}{\partial^{+4}} (\partial^+, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) + \left(\frac{1}{\partial^+}, \frac{1}{\partial^{+4}} (\partial^{+2}, \frac{\partial^2 \bar{d}^4}{\partial^{+2}}) \right) \right] .$$

If $M + 4 > 0$, then these terms must vanish by themselves, thus leading to the vanishing coefficient

$$M - 2(C + D) + 5 = 0 . \tag{F.7}$$

If $M + 4 \leq 0$, on the other hand, these terms are no longer singular and thus they do not have to vanish by themselves.

We then need to investigate terms along $\frac{\partial^2 \bar{d}^4}{\partial^{+n}}$ singular structure. Assuming that $M > -4$, we proceed with the terms of $\frac{\partial^2 \bar{d}^4}{\partial^{+n}}$ singular structure. After a little calculation with (F.7), we find that the most singular terms are of $\frac{\partial^2 \bar{d}^4}{\partial^{+3}}$, and given by

$$\begin{aligned}
& (B - \frac{7}{2}) \frac{1}{\partial^{+(A+1)}} \left(\partial^{+3} [\partial^{+(B-3)} \varphi^b \partial^{+(C-M-2)} \varphi^c] \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \right) \\
& + (M - C - 3D + 5) \frac{1}{\partial^{+(A+1)}} \left(\partial^{+3} [\partial^{+(B-2)} \varphi^b \partial^{+(C-M-3)} \varphi^c] \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \right) , \tag{F.8}
\end{aligned}$$

where the first term comes from only the \mathcal{F} -terms while the last term from both the \mathcal{F} - and \mathcal{G} -terms in (F.6). Since these terms are the most singular, they must either vanish by themselves or be canceled with (F.4). There are three possibilities for such cancelations:

- The first term in (F.4) cancels the first term of (F.8), and the remaining term in (F.8) vanishes by itself due to either symmetry or a vanishing coefficient. The necessary condition for this is to have the same powers of ∂^+ 's on the “first” and “third” superfields

$$D - 3 = B - 6 \quad \longrightarrow \quad B = D + 3 , \quad (\text{F.9})$$

and some symmetry in f^a_{bcd} allowing the interchange of the indices b and d . By comparing (F.8) with (F.4), we get

$$\begin{aligned} & (B - \frac{7}{2}) \frac{f^a_{bcd}}{\partial^{+(A+1)}} \left(\partial^{+3} [\partial^{+D} \varphi^b \partial^{+(C-M-2)} \varphi^c] \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \right) \\ &= (B - \frac{7}{2}) \frac{f^a_{dcb}}{\partial^{+(A+1)}} \left(\partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \partial^{+(1-M)} [\partial^{+C} \varphi^c \partial^{+D} \varphi^b] \right) , \end{aligned} \quad (\text{F.10})$$

up to an overall sign, which implies that $M = -2$. It follows from (F.7) that

$$2(C + D) = 3 . \quad (\text{F.11})$$

Now we consider the remaining term, the second term in (F.8), which must vanish either by the vanishing coefficient $M - C - 3D + 5$, or by the antisymmetry of $f^a_{[bc]d}$ requiring $B - 2 = C - M - 3$. The vanishing coefficient, which gives $C + 3D = 3$, cannot lead to a solution because together with (F.11), it leads to $C = D$, which then contradicts the assumption $C > D$. The other case with antisymmetry in b and c requires $B = C + 1$, which leads to $4(D + 1) = 3$ to meet (F.9) and (F.11), but then this yields fractional powers. Hence, both cases do not yield integer-valued solutions.

- The second possibility comes from the observation that the first terms of (F.4) and (F.8) have the same coefficients, which allows us to consider the possibility that these two terms can add up, to cancel against the second term of (F.8). This also needs the condition (F.9), but requires that

$$\begin{aligned} & (M - C - 3D + 5) \frac{f^a_{bcd}}{\partial^{+(A+1)}} \left(\partial^{+3} [\partial^{+(D+1)} \varphi^b \partial^{+(C-M-3)} \varphi^c] \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \right) \\ &= (2B - 7) \frac{f^a_{dcb}}{\partial^{+(A+1)}} \left(\partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \partial^{+(1-M)} [\partial^{+C} \varphi^c \partial^{+D} \varphi^b] \right) , \end{aligned} \quad (\text{F.12})$$

up to an overall sign. Notice that the powers of ∂^+ 's on φ^b 's on both side are different, which makes us to impose further condition

$$D = C - M - 2, \quad f^a{}_{bcd} = -f^a{}_{cbd} \quad (\text{F.13})$$

so that

$$\begin{aligned} & \frac{f^a{}_{bcd}}{\partial^{+(A+1)}} \left(\partial^{+3} [\partial^{+(D+1)} \varphi^b \partial^{+(C-M-3)} \varphi^c] \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \right) \\ &= \frac{f^a{}_{bcd}}{\partial^{+(A+1)}} \left(\partial^{+4} [\partial^{+D} \varphi^b \partial^{+(C-M-3)} \varphi^c] \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \right). \end{aligned} \quad (\text{F.14})$$

By comparing this to (F.12), we find that $M = -3$, and thus $D = C + 1$ which is in contradiction with the assumption $C > D$.

- The last possibility goes as follows. The first term in (F.8) vanishes due to a symmetry and the second term has the vanishing coefficient, regardless of the first term in (F.4), thus requiring

$$B - 3 = C - M - 2, \quad \text{and} \quad M - C - 3D + 5 = 0 \quad \rightarrow \quad 3D + B = 6, \quad (\text{F.15})$$

as well as (F.9). These conditions then lead to another fractional solution $D = 3/4$. Thus, this possibility does not yield integer-valued solutions.

We note that one might believe that there is another possibility that the second term of (F.8) can be canceled by the first term of (F.4), but this case cannot lead to integer valued solutions, so we neglect this possibility.

For $M + 4 < 0$, (F.7) is not required, but the leading singular terms are, however, still of a similar form as (F.8)

$$\begin{aligned} & (B + 2C - \frac{11}{2}) \frac{1}{\partial^{+(A+1)}} \left(\partial^{+3} [\partial^{+(B-3)} \varphi^b \partial^{+(C-M-2)} \varphi^c] \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \right) \\ &+ (M - C - 3D + 5) \frac{1}{\partial^{+(A+1)}} \left(\partial^{+3} [\partial^{+(B-2)} \varphi^b \partial^{+(C-M-3)} \varphi^c] \partial^2 \bar{d}^4 \partial^{+(D-3)} \varphi^d \right). \end{aligned} \quad (\text{F.16})$$

Calculations for this case are similar to those for the odd case and there are no integer-valued solutions.

Therefore, we have explored all possible cancelations for (F.8), and showed that there are no integer-valued exponents that make (F.8) cancel out or vanish. Thus we conclude that there are no solutions for the even case except the trivial one, (4.65).

G Bringing $\delta_{\bar{\epsilon}Q}^{int} \varphi^a$ to the BLG form

The conjugated ansatz (4.34) in the odd case is

$$\delta_{\bar{\epsilon}Q}^{int} \varphi^a = i\bar{\epsilon}_m \mathcal{S}^{-1} \frac{1}{3!} \varepsilon^{ijkl} \left(\frac{\partial}{\partial \eta^{ijk}} \frac{\partial}{\partial \zeta^l} \eta^m \mathcal{K}_{-\frac{1}{2}}^{a(0,\eta,\zeta)} - \frac{\partial}{\partial \eta^i} \frac{\partial}{\partial \zeta^{jkl}} \eta^m \mathcal{K}_{\frac{1}{2}}^{a(0,\eta,\zeta)} \right) \Big|_{\eta=\zeta=0}. \quad (\text{G.1})$$

Upon differentiating η^m it becomes

$$\delta_{\bar{\epsilon}Q}^{int} \varphi^a = -\frac{i}{3!} \bar{\epsilon}_m \left(3\Psi_1^m - \Psi_2^m \right), \quad (\text{G.2})$$

where we defined

$$\Psi_1^m \equiv \varepsilon^{mijk} \frac{\partial}{\partial \eta^{ij}} \frac{\partial}{\partial \zeta^k} \mathcal{S}^{-1} \mathcal{K}_{-\frac{1}{2}}^{a(0,\eta,\zeta)} \Big|_{\eta=\zeta=0}, \quad \Psi_2^m \equiv \varepsilon^{mijk} \frac{\partial}{\partial \zeta^{ijk}} \mathcal{S}^{-1} \mathcal{K}_{\frac{1}{2}}^{a(0,\eta,\zeta)} \Big|_{\eta=\zeta=0}. \quad (\text{G.3})$$

With the BLG values for the exponents (4.67), we have

$$\begin{aligned} \mathcal{S}^{-1} \mathcal{K}_{-\frac{1}{2}}^{a(0,\eta,\zeta)} &= f^a{}_{bcd} \frac{1}{\partial^{+2}} \left(\partial^{+2} E_\eta \varphi^b \cdot \partial^+ E_{-\eta} (\partial^+ E_\zeta \varphi^c \cdot \partial^+ E_{-\zeta} \varphi^d) \right), \\ \mathcal{S}^{-1} \mathcal{K}_{\frac{1}{2}}^{a(0,\eta,\zeta)} &= f^a{}_{bcd} \frac{1}{\partial^+} \left(\partial^+ E_\eta \varphi^b \cdot \frac{1}{\partial^+} E_{-\eta} (\partial^{+2} E_\zeta \varphi^c \cdot \partial^{+2} E_{-\zeta} \varphi^d) \right). \end{aligned} \quad (\text{G.4})$$

Differentiating and using the $[cd]$ antisymmetry, $f^a{}_{bcd} = -f^a{}_{bdc}$, we find

$$\begin{aligned} \Psi_1^m &= 2f^a{}_{bcd} \varepsilon^{mijk} \frac{1}{\partial^{+2}} \left[\bar{d}_{ij} \varphi^b \cdot \partial^+ (\bar{d}_k \varphi^c \cdot \partial^+ \varphi^d) + \partial^{+2} \varphi^b \cdot \frac{1}{\partial^+} \bar{d}_{ij} (\bar{d}_k \varphi^c \cdot \partial^+ \varphi^d), \right. \\ &\quad \left. - 2\partial^+ \bar{d}_i \varphi^b \cdot \bar{d}_j (\bar{d}_k \varphi^c \cdot \partial^+ \varphi^d) \right] \\ \Psi_2^m &= 2f^a{}_{bcd} \varepsilon^{mijk} \frac{1}{\partial^+} \left\{ \partial^+ \varphi^b \cdot \frac{1}{\partial^+} \left[\frac{1}{\partial^+} \bar{d}_{ijk} \varphi^c \cdot \partial^{+2} \varphi^d - 3\bar{d}_{ij} \varphi^c \cdot \partial^+ \bar{d}_k \varphi^d \right] \right\}. \end{aligned} \quad (\text{G.5})$$

Using the $[bd]$ symmetry, $f^a{}_{bcd} = -f^a{}_{dcb}$, we can rewrite Ψ_2^m as

$$\Psi_2^m = -2f^a{}_{bcd} \varepsilon^{mijk} \frac{1}{\partial^+} \left\{ \partial^+ \varphi^b \cdot \frac{1}{\partial^+} \left[\bar{d}_{ijk} \varphi^c \cdot \partial^+ \varphi^d + 3\bar{d}_{ij} \varphi^c \cdot \partial^+ \bar{d}_k \varphi^d \right] \right\}. \quad (\text{G.6})$$

Similarly, using total antisymmetry $f^a{}_{bcd} = f^a{}_{[bcd]}$, we find that

$$\Psi_1^m = -\Psi_2^m \quad (\text{G.7})$$

and therefore

$$\delta_{\bar{\epsilon}Q}^{int} \varphi^a = -\frac{4i}{3} \bar{\epsilon}_m f^a{}_{bcd} \varepsilon^{mijk} \frac{1}{\partial^+} \left\{ \partial^+ \varphi^b \cdot \frac{1}{\partial^+} \left[\bar{d}_{ijk} \varphi^c \cdot \partial^+ \varphi^d + 3\bar{d}_{ij} \varphi^c \cdot \partial^+ \bar{d}_k \varphi^d \right] \right\}. \quad (\text{G.8})$$

Finally, using the following two forms of the inside-out constraint (2.3)

$$\frac{1}{3!} \varepsilon^{mijk} \bar{d}_{ijk} \varphi^c = (i\sqrt{2}\partial^+) d^m \bar{\varphi}^c, \quad \frac{1}{2} \varepsilon^{mijk} \bar{d}_{ij} \varphi^c = d^{mk} \bar{\varphi}^c \quad (\text{G.9})$$

we arrive at the result given in (4.69).

H “C-only” projection of the Hamiltonian

First, we rewrite equation (5.9) as

$$H^{(1)} = \frac{8i}{2\sqrt{2}} \int d^3x \left(2X - i\sqrt{2}Y \right) + c.c., \quad (\text{H.1})$$

where, using the equivalence of $\int d^4\theta d^4\bar{\theta}$ to projecting with $d^{[4]}\bar{d}^{[4]}$, we defined

$$\begin{aligned} X &\equiv f_{abcd} d^{[4]}\bar{d}^{[4]} \left\{ q^m \frac{\partial}{\partial^{+3}} \varphi^a \cdot \partial^+ \bar{\varphi}^b \cdot \frac{1}{\partial^+} (\partial^+ \bar{d}_m \varphi^c \cdot \partial^+ \bar{\varphi}^d) \right\} \Big|_{\theta=\bar{\theta}=0}, \\ Y &\equiv f_{abcd} d^{[4]}\bar{d}^{[4]} \left\{ q^m \frac{\partial}{\partial^{+3}} \varphi^a \cdot \partial^+ \bar{\varphi}^b \cdot \frac{1}{\partial^+} (\bar{d}_{mn} \varphi^c \cdot \partial^+ d^n \bar{\varphi}^d) \right\} \Big|_{\theta=\bar{\theta}=0}. \end{aligned} \quad (\text{H.2})$$

Using that

$$d^{[4]}\bar{d}^{[4]} = \frac{1}{4!4!} \varepsilon^{ijkl} \varepsilon_{rstu} d^{rstu} \bar{d}_{ijkl} \quad (\text{H.3})$$

and keeping only terms giving C^{mna} or $\bar{C}_{mn}^a = \frac{1}{2} \varepsilon_{mnkl} C^{kla} = (C^{mna})^*$ upon projection,

$$\bar{d}_{mn} \varphi^a \Big|_{\theta=\bar{\theta}=0} = -i\sqrt{2} \bar{C}_{mn}^a, \quad d^{mn} \bar{\varphi}^a \Big|_{\theta=\bar{\theta}=0} = -i\sqrt{2} C^{mna} \quad (\text{H.4})$$

we find the following formulae

$$\begin{aligned} d^{[4]}\bar{d}^{[4]} \left(q^m \varphi, \bar{\varphi}, \bar{d}_m \varphi, \bar{\varphi} \right) \Big|_{C\text{-only}} &= 2i\sqrt{2} \left(\partial^+ C^{mn}, \bar{C}_{ij}, \bar{C}_{mn}, C^{ij} \right), \\ d^{[4]}\bar{d}^{[4]} \left(q^m \varphi, \bar{\varphi}, \bar{d}_{mn} \varphi, d^n \bar{\varphi} \right) \Big|_{C\text{-only}} &= \\ &-8 \left(\partial^{+2} C^{mi}, \bar{C}_{ij}, \bar{C}_{mn}, C^{nj} \right) + 16 \left(\partial^+ C^{mi}, \bar{C}_{ij}, \partial^+ \bar{C}_{mn}, C^{nj} \right) \\ &-8 \left(\partial^+ C^{mn}, \bar{C}_{ij}, \partial^+ \bar{C}_{mn}, C^{ij} \right) - 4 \left(\partial^+ C^{mn}, \bar{C}_{ij}, \bar{C}_{mn}, \partial^+ C^{ij} \right) \end{aligned} \quad (\text{H.5})$$

Applying them to X and Y , inserting an extra ∂^+/∂^+ for C^{mia} in the first term in Y and partially integrating, we find ¹³

$$\begin{aligned} \mathcal{X} \equiv \left(\frac{i}{4\sqrt{2}} X + \frac{1}{8} Y \right) \Big|_{C\text{-only}} &= f_{abcd} \left\{ \frac{\partial}{\partial^{+2}} C^{mia} \cdot \partial^{+2} \bar{C}_{ij}^b \cdot \frac{1}{\partial^+} \left(\bar{C}_{mn}^c \partial^+ C^{nj} \right) \right. \\ &+ \frac{\partial}{\partial^{+2}} C^{mia} \cdot \partial^+ \bar{C}_{ij}^b \cdot \frac{1}{\partial^+} \left(3\partial^+ \bar{C}_{mn}^c \cdot \partial^+ C^{nj} + \bar{C}_{mn}^c \partial^{+2} C^{nj} \right) \\ &\left. - \frac{\partial}{\partial^{+2}} C^{mna} \cdot \partial^+ \bar{C}_{ij}^b \cdot \frac{1}{\partial^+} \left(\frac{3}{2} \partial^+ \bar{C}_{mn}^c \cdot \partial^+ C^{ij} + \frac{1}{2} \bar{C}_{mn}^c \partial^{+2} C^{ij} \right) \right\}. \end{aligned} \quad (\text{H.6})$$

Using the following identity

$$(C^{mi}, \bar{C}_{ij}, \bar{C}_{mn}, C^{nj}) - \frac{1}{2} (C^{mn}, \bar{C}_{ij}, \bar{C}_{mn}, C^{ij}) = -(C^{mn}, C^{ij}, \bar{C}_{mi}, \bar{C}_{nj}), \quad (\text{H.7})$$

¹³Total ∂ and ∂^+ derivatives can be dropped under $\int d^3x$ in (H.1).

which follows from $\varepsilon^{mnk[l}\varepsilon^{ijrs]}(\overline{C}_{kl}, \overline{C}_{ij}, \overline{C}_{mn}, \overline{C}_{rs}) = 0$ and $C^{mna}\overline{C}_{mn}^b = \overline{C}_{mn}^a C^{mnb}$, we obtain

$$\begin{aligned} \mathcal{X} &= f_{abcd} \left\{ \frac{\partial}{\partial^{+2}} C^{mia} \cdot \partial^{+2} \overline{C}_{ij}^b \cdot \frac{1}{\partial^{+}} \left(\overline{C}_{mn}^c \partial^{+} C^{njd} \right) \right. \\ &\quad \left. - \frac{\partial}{\partial^{+2}} C^{mna} \cdot \partial^{+} C^{ijb} \cdot \frac{1}{\partial^{+}} \left[3\partial^{+} \overline{C}_{mi}^c \cdot \partial^{+} \overline{C}_{nj}^d + \overline{C}_{mi}^c \partial^{+2} \overline{C}_{nj}^d \right] \right\}. \end{aligned} \quad (\text{H.8})$$

Using antisymmetry of C 's and $[cd]$ antisymmetry of f_{abcd} , we find that the first term in the square bracket vanishes, whereas the other term can be written as a total derivative. Therefore,

$$\mathcal{X} = f_{abcd} \left\{ \frac{\partial}{\partial^{+2}} C^{mia} \cdot \partial^{+2} \overline{C}_{ij}^b \cdot \frac{1}{\partial^{+}} \left(\overline{C}_{mn}^c \partial^{+} C^{njd} \right) - \frac{\partial}{\partial^{+2}} C^{mna} \cdot \partial^{+} C^{ijb} \cdot \overline{C}_{mi}^c \partial^{+} \overline{C}_{nj}^d \right\}. \quad (\text{H.9})$$

Partially integrating one ∂^{+} on \overline{C}_{ij}^b and using (H.7) gives

$$\begin{aligned} \mathcal{X} &= f_{abcd} \left\{ - \frac{\partial}{\partial^{+}} C^{mia} \cdot \partial^{+} \overline{C}_{ij}^b \cdot \frac{1}{\partial^{+}} \left(\overline{C}_{mn}^c \partial^{+} C^{njd} \right) \right. \\ &\quad \left. - \frac{1}{2} \frac{\partial}{\partial^{+2}} C^{mna} \cdot \partial^{+} \overline{C}_{ij}^b \cdot \overline{C}_{mn}^c \partial^{+} C^{ijd} \right\}. \end{aligned} \quad (\text{H.10})$$

The second term vanishes thanks to $[bd]$ antisymmetry of f_{abcd} . Partially integrating ∂^{+} on \overline{C}_{ij}^b in the remaining term, we find

$$\mathcal{X} = f_{abcd} \partial C^{mia} \cdot \overline{C}_{ij}^b \frac{1}{\partial^{+}} \left(\overline{C}_{mn}^c \partial^{+} C^{njd} \right), \quad (\text{H.11})$$

where the other term vanished thanks to $[bc]$ antisymmetry of f_{abcd} . Applying (H.7) in the following form

$$(C^{mi}, \overline{C}_{ij}, \overline{C}_{mn}, C^{nj}) = \frac{1}{2} (C^{ij}, \overline{C}_{ij}, \overline{C}_{mn}, C^{mn}) - (\overline{C}_{mi}, C^{ij}, \overline{C}_{nj}, C^{mn}) \quad (\text{H.12})$$

we obtain

$$\mathcal{X} = f_{abcd} \left\{ \frac{1}{2} \partial C^{ija} \cdot \overline{C}_{ij}^b \frac{1}{\partial^{+}} \left(\overline{C}_{mn}^c \partial^{+} C^{mnd} \right) - \partial \overline{C}_{mi}^a \cdot C^{ijb} \frac{1}{\partial^{+}} \left(\overline{C}_{nj}^c \partial^{+} C^{mnd} \right) \right\}. \quad (\text{H.13})$$

On another hand, complex conjugating \mathcal{X} as given in (H.11), we find

$$\mathcal{X}^* = -f_{abcd} \partial \overline{C}_{mi}^a \cdot C^{ijb} \frac{1}{\partial^{+}} (\partial^{+} \overline{C}_{nj}^c \cdot C^{mnd}), \quad (\text{H.14})$$

where we also used $[cd]$ antisymmetry of f_{abcd} . Adding now (H.13) and (H.14), we find

$$\mathcal{X} + \mathcal{X}^* = \frac{1}{2} f_{abcd} \partial C^{ija} \cdot \overline{C}_{ij}^b \frac{1}{\partial^{+}} \left(\overline{C}_{mn}^c \partial^{+} C^{mnd} \right), \quad (\text{H.15})$$

where the other two terms cancel thanks to $[bd]$ antisymmetry of f_{abcd} . It then follows that

$$H^{(1)}|_{C\text{-only}} = 32 \int d^3x (\mathcal{X} + \mathcal{X}^*) = -16 f_{abcd} \int d^3x (\overline{C}_{ij}^a \partial C^{ijb}) \frac{1}{\partial^{+}} (\overline{C}_{mn}^c \partial^{+} C^{mnd}), \quad (\text{H.16})$$

where we used $[ab]$ antisymmetry of f_{abcd} . This proves that (5.9) has the “C-only” part as given in (5.12). On another hand, we have

$$\begin{aligned}
d^{[4]}\bar{d}^{[4]}\left\{(\varphi^a\partial\varphi^b)\frac{1}{\partial^+}(\bar{\varphi}^c\partial^+\bar{\varphi}^d)\right\}|_{C\text{-only}} &= \frac{6 \cdot 6}{4!4!}\varepsilon^{ijkl}\varepsilon_{rstu}(\bar{d}_{ij}\varphi^a \cdot \partial\bar{d}_{kl}\varphi^b)\frac{1}{\partial^+}(d^{rs}\bar{\varphi}^c \cdot \partial^+d^{tu}\bar{\varphi}^d)|_{\theta=\bar{\theta}=0} \\
&= \frac{(-i\sqrt{2})^4}{16}\varepsilon^{ijkl}\varepsilon_{rstu}(\bar{C}_{ij}^a\partial\bar{C}_{kl}^b)\frac{1}{\partial^+}(C^{rsc}\partial^+C^{tud}) \\
&= (\bar{C}_{ij}^a\partial C^{ijb})\frac{1}{\partial^+}(\bar{C}_{mn}^c\partial^+C^{mnd})
\end{aligned} \tag{H.17}$$

and, as this result is real, this proves that (5.13) also has the “C-only” part as given in (5.12).

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